



**University of
Zurich^{UZH}**

**Zurich Open Repository and
Archive**

University of Zurich
University Library
Strickhofstrasse 39
CH-8057 Zurich
www.zora.uzh.ch

Year: 2013

Random permutation matrices under the generalized Ewens measure

Hughes, Christopher ; Najnudel, Joseph ; Nikeghbali, Ashkan ; Zeindler, Dirk

Abstract: We consider a generalization of the Ewens measure for the symmetric group, calculating moments of the characteristic polynomial and similar multiplicative statistics. In addition, we study the asymptotic behavior of linear statistics (such as the trace of a permutation matrix or of a wreath product) under this new measure. © 2013 Institute of Mathematical Statistics.

DOI: <https://doi.org/10.1214/12-AAP862>

Posted at the Zurich Open Repository and Archive, University of Zurich

ZORA URL: <https://doi.org/10.5167/uzh-86954>

Journal Article

Published Version

Originally published at:

Hughes, Christopher; Najnudel, Joseph; Nikeghbali, Ashkan; Zeindler, Dirk (2013). Random permutation matrices under the generalized Ewens measure. *Annals of Applied Probability*, 23(3):987-1024.

DOI: <https://doi.org/10.1214/12-AAP862>

RANDOM PERMUTATION MATRICES UNDER THE GENERALIZED EWENS MEASURE

BY CHRISTOPHER HUGHES, JOSEPH NAJNUDEL,
ASHKAN NIKEGHBALI AND DIRK ZEINDLER¹

*University of York, Universität Zürich, Universität Zürich
and Universität Bielefeld*

We consider a generalization of the Ewens measure for the symmetric group, calculating moments of the characteristic polynomial and similar multiplicative statistics. In addition, we study the asymptotic behavior of linear statistics (such as the trace of a permutation matrix or of a wreath product) under this new measure.

1. Introduction. The trace of a matrix is one of the most natural additive class functions associated to the spectra of a matrix. Traces of unitary matrices chosen randomly with Haar measure have been much studied, for example by Diaconis and Shahshahani [8], Diaconis and Evans [6] and Rains [24] using methods from representation theory, by Diaconis and Gamburd [7] using combinatorics and using methods from mathematical physics by Haake et al. [11].

Another natural class function, this time multiplicative, is the characteristic polynomial, and the distribution of characteristic polynomials of random unitary matrices has been studied by many authors, including Keating and Snaith [17] and Hughes, Keating and O’Connell [13].

From the characteristic polynomial one can find the number of eigenvalues lying in a certain arc (since the underlying matrix is unitary, all the eigenvalues lie on the unit circle). The problem of studying the number of eigenvalues lying in an arc was studied by Rains [24] and Wieand [26] who found a very interesting correlation structure when multiple arcs were considered, and Hughes, Keating and O’Connell [13] who made the connection with characteristic polynomials.

One of the reasons for such an extensive study into random unitary matrices and their spectra is that the statistical distribution of the eigenvalues is expected to have the same behavior as the zeros of the Riemann zeta function (see Montgomery [21] and Keating and Snaith [17]).

The statistics of the distribution of spectra of infinite subgroups of the unitary group, such as the symplectic and orthogonal groups, are also expected to model

Received November 2011; revised March 2012.

¹Supported by the Swiss National Science Foundation (SNF).

MSC2010 subject classifications. 60B15.

Key words and phrases. Symmetric group, generalized Ewens measure, random permutation matrix, characteristic polynomial, multiplicative class functions, traces, linear statistics, limit theorems.

families of other L -functions (Keating and Snaith [16]), and the distribution of traces for these groups have been studied, frequently in the same papers.

However, the statistics of the spectra of finite subgroups of the unitary group, such as the permutation group, is not so well studied, though there are many results known.

Wieand [25] studied the number of eigenvalues of a uniformly random permutation matrix lying in a fixed arc, and Hambly et al. [12] found corresponding results for the characteristic polynomial, making the same connection between the characteristic polynomial and the counting function of eigenvalues.

In all these cases, the permutation matrices were chosen with uniform measure and the results were similar to those found for the full unitary group with Haar measure. However, there were some significant differences primarily stemming from the fact that the full unitary group is rotation invariant, so the characteristic polynomial is isotropic. The group of permutation matrix is clearly not rotation invariant, and the distribution of the characteristic polynomial depends weakly on the angle of the parameter. Most results require the angle to have the form $2\pi\tau$ with τ irrational and of finite type. (It is worth pointing out that those angles which are not of this type, have Hausdorff dimension zero). More recently Ben Arous and Dang [2] have extended some of the results of Wieand to more general measures together with new observations very specific to permutation matrices. In particular, they prove that the fluctuations of smooth linear statistics (with bounded variance) of random permutation matrices sampled under the Ewens measure are asymptotically non-Gaussian but infinitely divisible.

A permutation matrix has the advantage that the eigenvalues are determined by the cycle-type $\lambda = (\lambda_1, \dots, \lambda_\ell)$ of the corresponding permutation. This allows us to write down in many situations explicit expression for the studied object. For example, the characteristic polynomial of an $N \times N$ permutation matrix M is

$$\begin{aligned} Z_N(x) = \det(I - xM) &= \prod_{j=1}^N (1 - xe^{i\alpha_j}) = \prod_{m=1}^{\ell(\lambda)} (1 - x^{\lambda_m}) \\ (1.1) \qquad &= \prod_{k=1}^N (1 - x^k)^{C_k}, \end{aligned}$$

where $e^{i\alpha_1}, \dots, e^{i\alpha_N}$ are the eigenvalues of M and C_k is the number of cycles of length k (i.e., the number of j such that $\lambda_j = k$).

Equation (1.1) has been used by Dehay and Zeindler [5] to introduce multiplicative class functions (i.e., invariant under conjugation) associated to function f . These functions have the form

$$(1.2) \qquad W^N(f)(x) = \prod_{k=1}^N f(x^k)^{C_k}$$

and generalize the characteristic polynomial, which is the case $f(x) = (1 - x)$. This nice generalization is not possible for the unitary group because the eigenvalues lack the structure of eigenvalues of permutation matrices that allows (1.1) to hold. The most natural analogue for unitary matrices is Heine's identity which connects Haar averages of multiplicative class functions with determinants of Toeplitz matrices.

Equation (1.1) can also be used to express linear statistics or traces of functions of random permutation matrices in terms of the cycle counts. More precisely, if we identify a random permutation matrix M with the permutation σ it represents, we have the following definition.

DEFINITION 1.1. Let $F : S^1 \rightarrow \mathbb{C}$ be given. We then define the trace of F to be the function $\text{Tr}(F) : \mathfrak{S}_N \rightarrow \mathbb{C}$ with

$$(1.3) \quad \text{Tr}(F)(\sigma) := \sum_{k=1}^N F(\omega_k),$$

where $(\omega_k)_{k=1}^N$ are the eigenvalues of σ with multiplicity.

Observe that when $F(x) = x^d$, we have $\text{Tr}(F)(\sigma) = \text{Tr}(\sigma^d)$, and this justifies the use of the terminology trace. The trace of a function is also referred to as a linear statistic on \mathfrak{S}_N .

LEMMA 1.2. Let $F : S^1 \rightarrow \mathbb{C}$ and $\sigma \in \mathfrak{S}_N$ with cycle type λ be given. We then have

$$(1.4) \quad \text{Tr}(F)(\sigma) = \sum_{k=1}^N k C_k \Delta_k(F)$$

with $\Delta_k(F) := \frac{1}{k} \sum_{m=1}^k F(e^{2\pi i m/k})$.

PROOF. This follows immediately from equation (1.1). \square

The reason why the expressions in (1.1) and (1.2) are useful is that many things are known about the cycle counts C_k . For example, the cycle counts C_k converge weakly to independent Poisson random variables P_k , with mean $1/k$. Also useful in this context is the Feller coupling, since in many situations this allows one to replace C_k by P_k . Several details on the cycle counts and the Feller coupling can be found in the book by Arratia, Barbour and Tavaré [1].

These results all concern the uniform measure, where each permutation has weight $1/N!$, or the Ewens measure, where the probability is proportional to the total number of cycles, $\theta^{\ell(\lambda)}/(N!h_N)$, with h_N the required normalization constant (the case $\theta = 1$ corresponding to the uniform measure). A common ingredient in all the above cited works is the use of the Feller coupling (details can again be

found in the book by Arratia, Barbour and Tavaré [1]) and some improvements on the known bounds for the approximation given by this coupling (see [2], Section 4).

In recent years, there have been many works in random matrix theory aimed at understanding how much the spectral properties of random matrices depend on the probability distributions of its entries. Here a similar question translates into how are the linear and multiplicative (i.e., multiplicative class functions) statistics affected if one considers more general probability distributions than the Ewens measure on the symmetric group? The Ewens measure can be naturally generalized to a weighted probability measure which assigns to the permutation matrix M (i.e., to the associated permutation) the weight

$$\frac{1}{N!h_N} \prod_{k=1}^N \theta_k^{C_k},$$

where h_N is a normalization constant. The Ewens measure corresponds to the special case where $\theta_k = \theta$ is a constant. This measure has recently appeared in mathematical physics models (see, e.g., [3] and [9]) and one has only recently started to gain insight into the cycle structures of such random permutations. One major obstacle with such measures is that there exists nothing such as the Feller coupling and therefore the classical probabilistic arguments do not apply here. In a recent work, Nikeghbali and Zeindler [23] propose a new approach based on combinatorial arguments and singularity analysis of generating functions to obtain asymptotic expansions for the characteristic functions of the C_k 's as well as the total number of cycles, thus extending the classical limit theorems (and some distributional approximations) for the cycle structures of random permutations under the Ewens measure. In this paper we shall use the methods introduced in [23], namely, some combinatorial lemmas, generating series and singularity analysis to study linear and multiplicative statistics for random permutation matrices under the general weighted probability measure. In fact, we shall consider the more general random matrix model obtained from the wreath product $S^1 \wr \mathfrak{S}_N$ (see, e.g., [27]); this amounts to replacing the 1's in the permutation matrices by independent random variables taking values in the unit circle S^1 . The distribution of eigenvalues of such matrices (alongside other generalizations) has been studied previously by Najnudel and Nikeghbali [22]. It should be noted that many groups closely related to \mathfrak{S}_N exhibit such matrices, for instance, the Weyl group of $\mathrm{SO}(2N)$.

More precisely this paper is organized as follows.

In Section 2 we fix some notation and terminology, recall some useful combinatorial lemmas together with some results of Hwang (and some slight extensions) on singularity analysis of generating functions. In particular, we shall introduce two relevant classes of generating functions according to their behavior near singularities on the circle of convergence. In this article, we shall state our theorems for random matrices under the generalized Ewens measures for which the generating series of $(\theta_k)_{k \geq 1}$ is in one of these two classes.

In Section 3 we study the multiplicative class functions associated to a function f and obtain the asymptotic behavior of the joint moments. In particular, we extend earlier results of [5, 28] and of [12] on the characteristic polynomial of uniformly chosen random permutation matrices.

In Section 4 we focus both on the traces of powers and powers of traces which are classical statistics in random matrix theory. In fact we prove more generally that the fluctuations of the linear statistics for Laurent polynomials are asymptotically infinitely divisible (they converge in law to an infinite weighted sum of independent Poisson variables). We also establish the convergence of the integer moments of linear statistics for functions of bounded variation together with the rate of convergence.

In Section 5 we consider the more general model consisting of the wreath product $S^1 \wr \mathfrak{S}_N$ and study the linear statistics for general functions F in (1.3). In such models, the 1's in the permutation matrix are replaced with $(z_j)_{1 \leq j \leq N}$ which are i.i.d. random variables taking their values on the unit circle S^1 . In this framework, Lemma 1.2 can be naturally extended (see Lemma 5.1) and the quantity

$$(1.5) \quad \Delta_k(F, z) := \frac{1}{k} \sum_{\omega^k=z} F(\omega)$$

naturally appears in our technical conditions. Under some conditions on rate of convergence to 0 of the L^1 -norm of $\Delta_k(F, z)$, and some assumptions on the singularities of the generating series of $(\theta_k)_{k \geq 1}$, we are able to compute the asymptotics of the characteristic function of $\text{Tr}(F)$ with a good error term. From these asymptotics we are able to compute the fluctuations of $\text{Tr}(F)$. We also translate our conditions in terms of the Fourier coefficients of F , where F has to be in some Sobolev space H^s .

In Section 6 we still work within the framework of the wreath product $S^1 \wr \mathfrak{S}_N$ and consider the case where the variance of $\text{Tr}(F)$ is diverging. This time we restrict ourselves to the Ewens measure since our methods do not seem to apply in this situation. Hence, we go back to probabilistic arguments (i.e., use the Feller coupling) to prove that under some technical conditions on F , the fluctuations are Gaussian. In fact, we essentially adapt the proof by Ben Arous and Dang [2] to these more general situations. Nonetheless our theorem is also slightly more general in that it applies to a larger class of functions F .

2. The generalized Ewens measure, generating series and singularity analysis. In this section we fix notation and we recall some facts about the symmetric group and generating functions, as well as the main results from singular analysis (we also provide some variants and extensions for the purpose of this paper). Our presentation closely follows [23].

2.1. *Some combinatorial lemmas and the generalized Ewens measure.* We present in this section some basic facts about \mathfrak{S}_N and then define the generalized Ewens measure. We give here only a very short overview and refer to [1] and [20] for more details.

2.1.1. *Conjugation classes and functions on \mathfrak{S}_N .* We first take a closer look at the conjugation classes of the symmetric group \mathfrak{S}_N (the group of all permutations of a set of N objects). We only need to consider the conjugation classes since all probability measures and functions considered in this paper are invariant under conjugation (i.e., they are class functions). It is well known that the conjugation classes of \mathfrak{S}_N can be parameterized with partitions of N .

DEFINITION 2.1. A partition λ is a sequence of nonnegative integers $\lambda_1 \geq \lambda_2 \geq \dots$ eventually trailing to 0's, usually omitted. The size of the partition is $|\lambda| := \sum_m \lambda_m$. We call λ a partition of N if $|\lambda| = N$, and this will be denoted by $\lambda \vdash N$. The length of λ is the largest ℓ such that $\lambda_\ell \neq 0$.

Let $\sigma \in \mathfrak{S}_N$ be arbitrary. We can write $\sigma = \sigma_1 \cdots \sigma_\ell$ with σ_m , $1 \leq m \leq \ell$, disjoint cycles of length λ_m . Since disjoint cycles commute, we can assume that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell$. We call the partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ the *cycle-type* of σ . We write \mathcal{C}_λ for the set of all $\sigma \in \mathfrak{S}_N$ with cycle type λ . One can now show that two elements $\sigma, \tau \in \mathfrak{S}_N$ are conjugate if and only if σ and τ have the same cycle-type and that \mathcal{C}_λ are the conjugation classes of \mathfrak{S}_N . Since this is well known, we omit the proof and refer the reader to [20] for more details.

DEFINITION 2.2. Let $\sigma \in \mathfrak{S}_N$ be given with cycle-type λ . The cycle numbers C_k are defined as

$$(2.1) \quad C_k = C_k(\sigma) := \#\{m : \lambda_m = k\}$$

and the total number of cycles $T(\sigma)$ is

$$(2.2) \quad T(\sigma) := \sum_{k=1}^N C_k.$$

The functions $C_k(\sigma)$ and $T(\sigma)$ depend only on the cycle type of σ are thus class functions. Clearly $T(\sigma)$ equals $\ell(\lambda)$, the length of the partition corresponding to σ .

All expectations in this paper have the form $\frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} u(\sigma)$ for a class function u . Since u is constant on conjugation classes, it is more natural to sum over all conjugation classes. We thus need to know the size of each conjugation class.

LEMMA 2.3. *We have*

$$(2.3) \quad |\mathcal{C}_\lambda| = \frac{|\mathfrak{S}_N|}{z_\lambda} \quad \text{with } z_\lambda := \prod_{k=1}^N k^{C_k} C_k!$$

with C_k defined in (2.1), and

$$(2.4) \quad \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} u(\sigma) = \sum_{\lambda \vdash N} \frac{1}{z_\lambda} u(C_\lambda)$$

for a class function $u : \mathfrak{S}_N \rightarrow \mathbb{C}$.

PROOF. The first part can be found in [20] or in [4], Chapter 39. The second part follows immediately from the first part. \square

2.1.2. *Definition of the generalized Ewens measures.* We now define the generalized Ewens measures.

DEFINITION 2.4. Let $\Theta = (\theta_k)_{k=1}^\infty$ be a sequence of strictly positive numbers. We define for $\sigma \in \mathfrak{S}_N$ with cycle-type λ

$$(2.5) \quad \mathbb{P}_\Theta[\sigma] := \frac{1}{h_N N!} \prod_{m=1}^{\ell(\lambda)} \theta_{\lambda_m} = \frac{1}{h_N N!} \prod_{k=1}^N \theta_k^{C_k(\sigma)}$$

with $h_N = h_N(\Theta)$ a normalization constant and $h_0 := 1$.

The second equality in (2.5) follows immediately from the definition of C_k (Definition 2.2). The uniform measure and the Ewens measure are special cases, with $\theta_k \equiv 1$ and $\theta_k \equiv \theta$ a constant, respectively.

We now introduce two generating functions closely related to \mathbb{P}_Θ :

$$(2.6) \quad g_\Theta(t) := \sum_{k=1}^{\infty} \frac{\theta_k}{k} t^k \quad \text{and} \quad G_\Theta(t) := \exp\left(\sum_{k=1}^{\infty} \frac{\theta_k}{k} t^k\right).$$

At the moment, $g_\Theta(t)$ and $G_\Theta(t)$ are just formal power series, however, we will see in Section 2.2 that

$$(2.7) \quad G_\Theta(t) = \sum_{N=0}^{\infty} h_N t^N,$$

where the h_N are given in Definition 2.4.

2.2. *Generating functions and singularity analysis.* The idea of generating functions is to encode information of a sequence into a formal power series.

DEFINITION 2.5. Let $(g_N)_{N \in \mathbb{N}}$ be a sequence of complex numbers and define the (ordinary) generating function of the sequence as the formal power series

$$(2.8) \quad G(t) = \sum_{N=0}^{\infty} g_N t^N.$$

We define $[t^N][G]$ to be the coefficient of t^N in $G(t)$, that is, $[t^N][G] := g_N$.

The reason why generating functions are useful is that it is often possible to compute the generating function without knowing g_N explicitly.

The main tool in this paper to calculate generating functions is the following lemma.

LEMMA 2.6. *Let $(a_m)_{m \in \mathbb{N}}$ be a sequence of complex numbers. Then*

$$(2.9) \quad \sum_{\lambda} \frac{1}{z_{\lambda}} \left(\prod_{m=1}^{\ell(\lambda)} a_{\lambda_m} \right) t^{|\lambda|} = \sum_{\lambda} \frac{1}{z_{\lambda}} \left(\prod_{k=1}^{\infty} (a_k t^k)^{C_k} \right) = \exp \left(\sum_{k=1}^{\infty} \frac{1}{k} a_k t^k \right)$$

with the same z_{λ} as in Lemma 2.3.

If any one of the sums in (2.9) is absolutely convergent, then so are the others.

PROOF. The first equality follows immediately from the definition of C_k . The proof of the second equality in (2.9) can be found in [20] or can be directly verified using the definitions of z_{λ} and the exponential function. The last statement follows with dominated convergence. \square

We now use this lemma to prove the identity given in (2.7). The constant h_N in (2.5) is chosen so that $\mathbb{P}_{\Theta}[\sigma]$ is a probability measure on \mathfrak{S}_N . It thus follows that

$$(2.10) \quad h_N = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \prod_{k=1}^N \theta_k^{C_k} = \sum_{\lambda \vdash N} \frac{1}{z_{\lambda}} \prod_{m=1}^{\ell(\lambda)} \theta_{\lambda_m}.$$

It now follows, with Lemma 2.6, that

$$(2.11) \quad \sum_{N=0}^{\infty} h_N t^N = \sum_{\lambda} \frac{1}{z_{\lambda}} t^{|\lambda|} \prod_{m=1}^{\ell(\lambda)} \theta_{\lambda_m} = \exp \left(\sum_{k=1}^{\infty} \frac{\theta_k}{k} t^k \right) = G_{\Theta}(t),$$

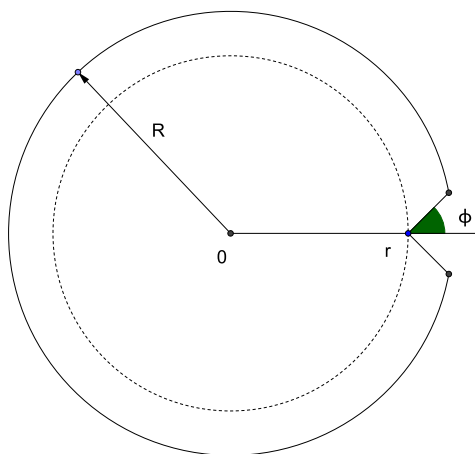
which proves (2.7).

COROLLARY 2.7. *In the special case of the Ewens measure, when θ_k is a constant θ , say, we have*

$$(2.12) \quad G_{\Theta}(t) = \sum_{N=0}^{\infty} h_N t^N = (1-t)^{-\theta}.$$

From this it immediately follows that $h_N = (-1)^N \binom{-\theta}{N} = \binom{N+\theta-1}{N}$.

Given a generating function $G(t)$, a natural question is: what is the coefficient of t^N and what is the asymptotic behavior of this coefficient as $N \rightarrow \infty$? If $G(t)$ is holomorphic near 0, then one can use Cauchy's integral formula to do this. But it turns out that it is often difficult to compute the integral exactly, but we will now see that one can nevertheless extract the asymptotic behavior of the coefficient when $G(t)$ has a special form.

FIG. 1. Illustration of $\Delta_0(r, R, \phi)$.

DEFINITION 2.8. Given $R > r$ and $0 < \phi < \frac{\pi}{2}$, let

$$(2.13) \quad \Delta_0 = \Delta_0(r, R, \phi) = \{z \in \mathbb{C} : |z| < R, z \neq r, |\arg(z - r)| > \phi\}.$$

The domain Δ_0 is illustrated in Figure 1.

DEFINITION 2.9. Let $r > 0$, $\vartheta \geq 0$ and a complex constant K be given. We say that a function $g(t)$ is in $\mathcal{F}(r, \vartheta, K)$ if there exists $R > r$ and $0 < \phi < \frac{\pi}{2}$ such that $g(t)$ is holomorphic in $\Delta_0(r, R, \phi)$, and

$$(2.14) \quad g(t) = \vartheta \log\left(\frac{1}{1 - t/r}\right) + K + O(t - r)$$

as $t \rightarrow r$ with $t \in \Delta_0(r, R, \phi)$.

The following theorem, proven by Hwang in [14], gives the asymptotic behavior of the coefficient of t^N for certain special generating functions.

THEOREM 2.10 (Hwang [14]). Let $g(t) \in \mathcal{F}(r, \vartheta, K)$, and let $S(t)$ be holomorphic in $|t| \leq r$. Set $G(t, w) = e^{wg(t)} S(t)$, then

$$(2.15) \quad [t^N][G(t, w)] = \frac{e^{Kw} N^{w\vartheta-1}}{r^N} \left(\frac{S(r)}{\Gamma(\vartheta w)} + O\left(\frac{1}{N}\right) \right)$$

uniformly for bounded complex w .

REMARK. The idea of the proof is to take a suitable Hankel contour and to estimate the integral over each piece. The details can be found in [14], Chapter 5.

REMARK. One can compute lower order error terms if one has more terms in the expansion of $g(t)$ near r .

As a first simple application of this result, we compute the asymptotic behavior of h_N for the generalized Ewens measure if $g_\Theta(t)$, as defined in (2.6), is in $\mathcal{F}(r, \vartheta, K)$.

LEMMA 2.11. *Let $g_\Theta(t) \in \mathcal{F}(r, \vartheta, K)$. We then have*

$$(2.16) \quad h_N = \frac{e^K N^{\vartheta-1}}{r^N} \left(\frac{1}{\Gamma(\vartheta)} + O\left(\frac{1}{N}\right) \right).$$

PROOF. We have proven in (2.11) that $\sum_{N=0}^{\infty} h_N t^N = \exp(g_\Theta(t))$. We thus can apply Theorem 2.10 with $g(t) = g_\Theta(t)$, $w = 1$ and $S(t) \equiv 1$. \square

REMARK. For the Ewens measure, when Θ is the constant sequence $(\theta)_{k=1}^{\infty}$, we have $g_\Theta(t) \in \mathcal{F}(1, \theta, 0)$ and thus $h_N = \frac{N^{\theta-1}}{\Gamma(\theta)} (1 + O(\frac{1}{N}))$. However, in this special case, one can do much more since h_N is known to equal $\binom{N+\theta-1}{N}$.

Essentially, one can think of Hwang's result as concerning functions with a solitary singularity at $t = r$. In Section 3.3 we will need a version of this theorem with multiple singularities that we now state.

DEFINITION 2.12. Let $\xi = (\xi_i)_{i=1}^d$ with $\xi_i \neq \xi_j$ for $i \neq j$, and with $|\xi_i| = r$ (so the ξ_i are distinct points lying on the circle of radius r). Let $R > r$ and let $0 < \phi < \frac{\pi}{2}$, then set

$$(2.17) \quad \begin{aligned} \Delta_d(r, R, \phi, \xi) \\ := \bigcap_{i=1}^d \{z \in \mathbb{C} : |z| < R, z \neq \xi_i, |\arg(z - \xi_i) - \arg(\xi_i)| > \phi\}. \end{aligned}$$

An example of a $\Delta_d(r, R, \phi, \xi)$ domain is given in Figure 2.

DEFINITION 2.13. Let $\vartheta = (\vartheta_i)_{i=1}^d$ and $\mathbf{K} = (K_i)_{i=1}^d$ be two sequences of complex numbers, and let $r > 0$. We say a function $g(t)$ is in $\mathcal{F}(r, \vartheta, \mathbf{K})$ if there exists $R > r$ and $0 < \phi < \frac{\pi}{2}$ such that $g(t)$ is holomorphic in $\Delta_d(r, R, \phi, \xi)$, and for each $i = 1, \dots, d$,

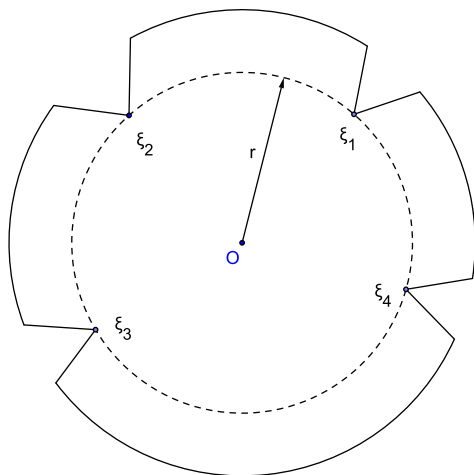
$$(2.18) \quad g(t) = \vartheta_i \log\left(\frac{1}{1 - t/\xi_i}\right) + K_i + O(t - \xi_i)$$

as $t \rightarrow \xi_i$ with $t \in \Delta_d(r, R, \phi, \xi)$.

Theorem 2.10 generalizes to the next theorem.

THEOREM 2.14. *Let $g \in \mathcal{F}(r, \vartheta, \mathbf{K})$, and let $S(t)$ be holomorphic in t for $|t| \leq r$. Set $G(t, w) = e^{wg(t)} S(t)$. We have*

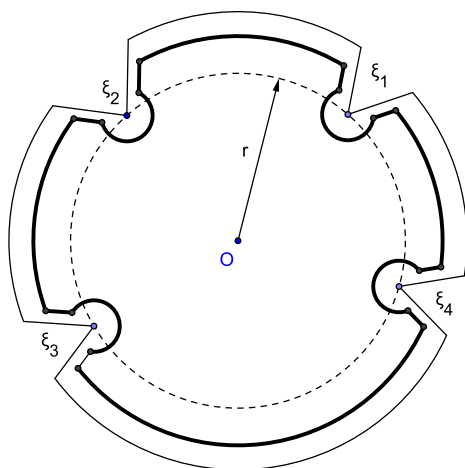
$$(2.19) \quad [t^N][G(t, w)] = \sum_{i=1}^d \frac{e^{K_i w} N^{w\vartheta_i-1}}{\xi_i^N} \left(\frac{S(\xi_i)}{\Gamma(\vartheta_i w)} + O\left(\frac{1}{N}\right) \right)$$

FIG. 2. Illustration of the domain $\Delta_d(r, R, \phi, \xi)$.

uniformly for bounded w .

SKETCH OF THE PROOF. The proof is a combination of the proof of a multiple singularities theorem in [10], Section VI.5, and the proof of Theorem 2.10. More precisely, we apply Cauchy's integral formula with the curve C illustrated in Figure 3, where the radius R of the great circle is chosen fix with $R > r$, while the radii of the small circles are $1/n$.

A straightforward computation then shows that the integral over this curve gives (2.19) and that the error terms are uniform for bounded w . \square

FIG. 3. The curve C .

In practice, the computation of the asymptotic behavior near the singularity is often very difficult, and it is not easy to prove whether a function $g(t)$ is in $\mathcal{F}(r, \vartheta, K)$ or not. An alternate approach is to combine singularity analysis with more elementary methods. The idea is to write $G = G_1 G_2$ in a way that we can apply singularity analysis on G_1 and can estimate the growth rate of $[t^N][G_2]$. One then can compute the coefficient $[t^N][G]$ directly and apply elementary analysis on it. This method is called the convolution method.

DEFINITION 2.15. Let $\vartheta \geq 0, r > 0, 0 < \gamma \leq 1$ be given. We say $g(t)$ is in $e\mathcal{F}(r, \vartheta, \gamma)$ if $g(t)$ is holomorphic in $|t| < r$ with

$$(2.20) \quad g(t) = \vartheta \log\left(\frac{1}{1-t/r}\right) + g_0(t)$$

and

$$(2.21) \quad [t^N][g_0] = O(r^{-N} N^{-1-\gamma})$$

as $N \rightarrow \infty$.

THEOREM 2.16 (Hwang [15]). Let $g(t) \in e\mathcal{F}(r, \vartheta, \gamma)$, and let $S(t)$ be holomorphic in $|t| \leq r$. Set $G(t, w) = e^{wg(t)} S(t)$, then

$$(2.22) \quad [t^N][G(t, w)] = \frac{e^{wg_0(r)} N^{w\vartheta-1}}{r^N} \frac{S(r)}{\Gamma(\vartheta w)} + R_N(w)$$

with

$$(2.23) \quad R_N(w) = \begin{cases} O\left(\frac{N^{\vartheta \operatorname{Re}(w)-1-\gamma} \log(N)}{r^N}\right), & \text{if } \operatorname{Re}(w) \geq 0, \\ O\left(\frac{N^{-1-\gamma}}{r^N}\right), & \text{if } \operatorname{Re}(w) < 0, \end{cases}$$

uniformly for bounded w .

This theorem is more general than Theorem 2.10, but the error terms are worse. As in Lemma 2.11, we can compute the asymptotic behavior of h_N in the case of the generalized Ewens measures when $g_\Theta(t) \in e\mathcal{F}(r, \vartheta, \gamma)$.

LEMMA 2.17. Assume that $g_\Theta(t) \in e\mathcal{F}(r, \vartheta, \gamma)$. We then have

$$(2.24) \quad h_N = \frac{e^{g_0(r)} N^{\vartheta-1}}{r^N \Gamma(\vartheta)} + O\left(\frac{N^{\vartheta-1-\gamma} \log N}{r^N}\right).$$

3. Moments of multiplicative class functions. We extend in this section the results of [5, 12] and [28] to the generalized Ewens measure \mathbb{P}_Θ . More precisely, we compute the asymptotic behavior of the moments of the characteristic polynomial $Z_N(x)$ and of multiplicative class functions $W^N(P)$ with respect to \mathbb{P}_Θ using the methods of generating functions and singularity analysis introduced in the previous section.

3.1. *Multiplicative class functions.* It is well known that \mathfrak{S}_N can be identified with the group of permutation matrices via

$$(3.1) \quad \sigma \mapsto (\delta_{i,\sigma(j)})_{1 \leq i,j \leq N}.$$

It is easy to see that this map is an injective group homomorphism. We thus do not distinguish between \mathfrak{S}_N and the group of permutation matrices and use for both the notation \mathfrak{S}_N . It will always be clear from the context if it is necessary to consider $\sigma \in \mathfrak{S}_N$ as a matrix.

DEFINITION 3.1. Let $x \in \mathbb{C}$ and $\sigma \in \mathfrak{S}_N$. The characteristic polynomial of σ is

$$(3.2) \quad Z_N(x) = Z_N(x)(\sigma) := \det(I_N - x\sigma).$$

It is a standard fact that the characteristic polynomial can be written in terms of the cycle type of σ .

LEMMA 3.2. Let $\sigma \in \mathfrak{S}_N$ be given with cycle type λ ; then

$$(3.3) \quad Z_N(x) = \prod_{m=1}^{\ell(\lambda)} (1 - x^{\lambda_m}),$$

with $\ell(\lambda)$ the length of the partition λ , which is the same as the number of cycles $T(\sigma)$.

PROOF. Since any permutation matrix is conjugate to a block matrix with each block corresponding to one of the cycles, and the characteristic polynomial factors over the blocks, it is sufficient to prove this result in the simple case of a one-cycle permutation, where it follows from a simple calculation. More explicit details can be found, for instance, in [28], Chapter 2.2. \square

Equation (3.3) shows that the spectrum of permutation matrix is uniquely determined by the cycle type. We use this as motivation to define multiplicative class functions on \mathfrak{S}_N .

DEFINITION 3.3. Let $P(x)$ be a polynomial in x . We then define the multiplicative class function associated to the polynomial P as

$$(3.4) \quad W^N(P)(x) = W^N(P)(x)(\sigma) := \prod_{m=1}^{\ell(\lambda)} P(x^{\lambda_m}).$$

For brevity, we simply call this a multiplicative class function.

It follows immediately that the characteristic polynomial is the multiplicative class function associated to the polynomial $P(x) = 1 - x$. The main difference between $Z_N(x)$ and $W^N(P)$ is that $W^N(P)$ is independent of the interpretation of \mathfrak{S}_N as matrices.

We now wish to obtain the asymptotic behavior of the moments

$$\mathbb{E}_\Theta[(W^N(P_1)(x_1))^{k_1}(W^N(P_2)(x_2))^{k_2}]$$

for $x_1 \neq x_2$. The easiest way to achieve this is to extend the definition of $W^N(P)$.

DEFINITION 3.4. Let $P(x_1, x_2)$ be a polynomial in the two variables x_1, x_2 . For $\sigma \in \mathfrak{S}_N$ with cycle type λ , we set

$$(3.5) \quad W^N(P)(x_1, x_2) = W^N(P)(x_1, x_2)(\sigma) := \prod_{m=1}^{\ell(\lambda)} P(x_1^{\lambda_m}, x_2^{\lambda_m}).$$

A simple computation using the definitions above shows

$$\begin{aligned} (W^N(P_1)(x_1))^{k_1}(W^N(P_2)(x_2))^{k_2} &= W^N(P_1^{k_1})(x_1)W^N(P_2^{k_2})(x_2) \\ &= W^N(P)(x_1, x_2) \end{aligned}$$

with $P(x_1, x_2) = P_1^{k_1}(x_1)P_2^{k_2}(x_2)$.

This shows that it is enough to consider $\mathbb{E}_\Theta[W^N(P)(x_1, x_2)]$.

REMARK. There is no restriction to the number of variables. All arguments used here work also for more than two variables. We restrict ourselves to two variables since this is enough to illustrate the general case.

3.2. Generating functions for W^N . In this section we compute the generating functions of the moments of multiplicative class functions.

THEOREM 3.5. Let P be a complex polynomial with

$$(3.6) \quad P(x_1, x_2) = \sum_{k_1, k_2=0}^{\infty} b_{k_1, k_2} x_1^{k_1} x_2^{k_2}.$$

We have as formal power series

$$(3.7) \quad \sum_{N=0}^{\infty} t^N h_N \mathbb{E}_\Theta[W^N(P)(x_1, x_2)] = \prod_{k_1, k_2=0}^{\infty} (G_\Theta(x_1^{k_1} x_2^{k_2} t))^{b_{k_1, k_2}},$$

where $G_\Theta(t)^b = \exp(b \cdot g_\Theta(t))$, and G_Θ and g_Θ are defined in (2.6).

PROOF. From (3.5) and (2.5) we have

$$(3.8) \quad \mathbb{E}_{\Theta}[W^N(P)(x_1, x_2)] = \frac{1}{h_N N!} \sum_{\sigma \in \mathfrak{S}_N} \prod_{m=1}^{\ell(\lambda)} \theta_{\lambda_m} P(x_1^{\lambda_m}, x_2^{\lambda_m})$$

and since $W^N(P)$ is a class function, we may use (2.4) to obtain

$$(3.9) \quad \mathbb{E}_{\Theta}[W^N(P)(x_1, x_2)] = \frac{1}{h_N} \sum_{\lambda \vdash N} \frac{1}{z_{\lambda}} \prod_{m=1}^{\ell(\lambda)} \theta_{\lambda_m} P(x_1^{\lambda_m}, x_2^{\lambda_m}).$$

We now compute the generating function of $h_N \mathbb{E}_{\Theta}[W^N(P)(x_1, x_2)]$ with the help of Lemma 2.6:

$$\begin{aligned} \sum_{N=0}^{\infty} t^N h_N \mathbb{E}[W^N(P)(x_1, x_2)] &= \sum_{N=0}^{\infty} t^N \sum_{\lambda \vdash N} \frac{1}{z_{\lambda}} \prod_{m=1}^{\ell(\lambda)} \theta_{\lambda_m} P(x_1^{\lambda_m}, x_2^{\lambda_m}) \\ &= \sum_{\lambda} \frac{1}{z_{\lambda}} t^{|\lambda|} \prod_{m=1}^{\ell(\lambda)} (\theta_{\lambda_m} P(x_1^{\lambda_m}, x_2^{\lambda_m})) \\ &= \exp\left(\sum_{m=1}^{\infty} \frac{\theta_m}{m} t^m P(x_1^m, x_2^m)\right) \\ &= \exp\left(\sum_{k_1, k_2=0}^{\infty} b_{k_1, k_2} \sum_{m=1}^{\infty} \frac{\theta_m}{m} (x_1^{k_1} x_2^{k_2} t)^m\right). \end{aligned}$$

Note that

$$(3.10) \quad \sum_{m=1}^{\infty} \frac{\theta_m}{m} (x_1^{k_1} x_2^{k_2} t)^m = g_{\Theta}(x_1^{k_1} x_2^{k_2} t),$$

where g_{Θ} is defined in (2.6), and hence,

$$(3.11) \quad \sum_{N=0}^{\infty} t^N h_N \mathbb{E}[W^N(P)(x_1, x_2)] = \prod_{k_1, k_2=0}^{\infty} (G_{\Theta}(x_1^{k_1} x_2^{k_2} t))^{b_{k_1, k_2}}$$

and this proves (3.7), as required. \square

REMARK. The requirement for P to be a polynomial is there to ensure absolute convergence, and clearly this condition can be considerably weakened (see [5], Sections 5, 6).

As an immediate consequence we have the following corollary.

COROLLARY 3.6. *Let $s_1, s_2 \in \mathbb{N}$ be given. Then*

$$(3.12) \quad \sum_{N=0}^{\infty} t^N h_N \mathbb{E}_{\Theta}[Z_N(x_1)^{s_1} Z_N(x_2)^{s_2}] \\ = \prod_{k_1=0}^{s_1} \prod_{k_2=0}^{s_2} (G_{\Theta}(x_1^{k_1} x_2^{k_2} t))^{(-1)^{k_1+k_2} \binom{s_1}{k_1} \binom{s_2}{k_2}}.$$

PROOF. We have

$$\begin{aligned} Z_N(x_1)^{s_1} Z_N(x_2)^{s_2} &= (W^N(1-x_1))^{s_1} (W^N(1-x_2))^{s_2} \\ &= W^N((1-x_1)^{s_1} (1-x_2)^{s_2}). \end{aligned}$$

The corollary now follows immediately by calculating the Taylor expansion of $(1-x_1)^{s_1} (1-x_2)^{s_2}$ near 0. \square

3.3. *Asymptotic behavior of the moments.* Combining the generating functions in Theorem 3.5 with the singularity analysis developed in Section 2.2, we compute the asymptotic behavior of $\mathbb{E}_{\Theta}[W^N(P)]$ as $N \rightarrow \infty$.

We have to distinguish between the cases $|x_i| < 1$ and $|x_i| = 1$. We consider here only $g_{\Theta}(t) \in \mathcal{F}(r, \vartheta, K)$. The results and computations for $g_{\Theta}(t) \in e\mathcal{F}(r, \vartheta, \gamma)$ are similar, with only minor differences in the error terms.

We first look at the asymptotic behavior inside the unit disc. We have the following theorem.

THEOREM 3.7. *Let P be as in Theorem 3.5, and let $x_1, x_2 \in \mathbb{C}$ be given with $\max\{|x_1|, |x_2|\} < 1$. Assume that $g_{\Theta}(t) \in \mathcal{F}(r, \vartheta, K)$, then*

$$(3.13) \quad \mathbb{E}[W^N(P)] = N^{\vartheta(b_{0,0}-1)} e^{K(b_{0,0}-1)} \left(E_1 + O\left(\frac{1}{N}\right) \right),$$

with

$$(3.14) \quad E_1 = E_1(x_1, x_2) = \frac{\Gamma(\vartheta)}{\Gamma(\vartheta b_{0,0})} \prod_{(k_1, k_2) \neq (0,0)} (G_{\Theta}(r x_1^{k_1} x_2^{k_2}))^{b_{k_1, k_2}}.$$

PROOF. Set

$$(3.15) \quad S(t) := \prod_{(k_1, k_2) \neq (0,0)} (G_{\Theta}(x_1^{k_1} x_2^{k_2} t))^{b_{k_1, k_2}}.$$

Since P is polynomial, the product is finite and there is no problem with convergence. The domain of holomorphicity of S is thus the intersection of the domains of holomorphicity of each factor. This shows that the function $S(t)$ is holomorphic for $|t| < r + \varepsilon$ for an $\varepsilon > 0$ since $\max\{|x_1|, |x_2|\} < 1$ and $G_{\Theta}(t)$ is holomorphic for $|t| < r$.

Separating the $k_1 = k_2 = 0$ term in (3.7) from the rest, we can write the generating function as

$$(3.16) \quad \sum_{N=0}^{\infty} t^N h_N \mathbb{E}_{\Theta}[W^N(P)(x_1, x_2)] = \exp(b_{0,0} \cdot g_{\Theta}(t)) S(t).$$

Applying Theorem 2.10, we get

$$h_N \mathbb{E}[W^N(P)] = N^{\vartheta b_{0,0}-1} e^{K b_{0,0}} \frac{1}{r^N} \left(\frac{S(r)}{\Gamma(\vartheta b_{0,0})} + O\left(\frac{1}{N}\right) \right).$$

Comparing $S(r)$ with E_1 , and using Lemma 2.11 to find the asymptotic behavior of h_N , proves the theorem. \square

As a special case, we get the asymptotic behavior of $\mathbb{E}_{\Theta}[Z_N^{s_1}(x_1)Z_N^{s_2}(x_2)]$ with respect to \mathbb{P}_{Θ} inside the unit disc.

COROLLARY 3.8. *Let $x_1, x_2 \in \mathbb{C}$ be given with $\max\{|x_1|, |x_2|\} < 1$ and let $s_1, s_2 \in \mathbb{N}$. We then have*

$$(3.17) \quad \begin{aligned} & \mathbb{E}_{\Theta}[Z_N^{s_1}(x_1)Z_N^{s_2}(x_2)] \\ &= \prod_{(k_1, k_2) \neq (0,0)} (G_{\Theta}(rx_1^{k_1}x_2^{k_2}))^{(-1)^{k_1+k_2} \binom{s_1}{k_1} \binom{s_2}{k_2}} + O\left(\frac{1}{N}\right). \end{aligned}$$

PROOF. This follows immediately from the fact that $Z_N(x) = W^N(1-x)(x)$ and that $(1-x_1)^{s_1}(1-x_2)^{s_2}$ evaluated at $x_1 = x_2 = 0$ is 1. \square

In particular, for the uniform measure ($\theta_k \equiv 1$ for all k) Corollary 2.7 gives $G_{\Theta}(t) = (1-t)^{-1}$ in which case we have

$$(3.18) \quad \begin{aligned} & \mathbb{E}[Z_N^{s_1}(x_1)Z_N^{s_2}(x_2)] \\ &= \prod_{(k_1, k_2) \neq (0,0)} (1 - x_1^{k_1}x_2^{k_2})^{-\binom{s_1}{k_1} \binom{s_2}{k_2} (-1)^{k_1+k_2}} + O\left(\frac{1}{N}\right). \end{aligned}$$

This shows that Corollary 3.8 agrees with [28], Theorem 2.13, in the uniform case.

The behavior on the unit disc is more complicated. The reason is that the generating function can have (for fixed x_1, x_2) more than one singularity on the circle of radius r . Another point that makes this case more laborious is the requirement to check whether some of the singularities of the factors on the right-hand side of (3.7) are equal. For simplicity, we assume that all singularities are distinct.

THEOREM 3.9. *Let P be as in Theorem 3.5, and let $x_1, x_2 \in \mathbb{C}$ be given with $|x_1| = |x_2| = 1$ and $x_1^{k_1}x_2^{k_2} \neq 1$ for all $(k_1, k_2) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$. Assume that $g_{\Theta}(t) \in$*

$\mathcal{F}(r, \vartheta, K)$, then

$$\begin{aligned} & \mathbb{E}_{\Theta}[W^N(P)] \\ &= \sum_{\substack{k_1, k_2 \\ b_{k_1, k_2} \neq 0}} E_2(k_1, k_2) N^{\vartheta(b_{k_1, k_2} - 1)} x_1^{Nk_1} x_2^{Nk_2} \left(\frac{\Gamma(\vartheta)}{\Gamma(\vartheta b_{k_1, k_2})} + O\left(\frac{1}{N}\right) \right) \end{aligned}$$

with

$$(3.19) \quad E_2(k_1, k_2) = e^{K(b_{k_1, k_2} - 1)} \prod_{(m_1, m_2) \neq (k_1, k_2)} (G_{\Theta}(rx_1^{m_1 - k_1} x_2^{m_2 - k_2}))^{b_{m_1, m_2}}.$$

PROOF. We define

$$(3.20) \quad F(t) = \sum_{k_1, k_2} b_{k_1, k_2} g_{\Theta}(x_1^{k_1} x_2^{k_2} t).$$

By (3.7) we see that $\exp(F(t))$ is the generating function of $h_N \mathbb{E}_{\Theta}[W^N(P)]$. We first take a look at the domain of holomorphicity of $F(t)$. We have by assumption that $g_{\Theta}(t)$ is holomorphic in $\Delta_0(r, R, \phi)$ for an $R > r$ and $0 < \phi < \frac{\pi}{2}$. This shows that $g_{\Theta}(x_1^{k_1} x_2^{k_2} t)$ is holomorphic for t in the domain $\Delta_1(r, R, \phi, rx_1^{-k_1} x_2^{-k_2})$ with Δ_1 as in Definition 2.12, and that F is holomorphic in

$$(3.21) \quad D := \bigcap_{\substack{k_1, k_2 \\ b_{k_1, k_2} \neq 0}} \Delta_1(r, R, \phi, rx_1^{-k_1} x_2^{-k_2}) = \Delta_d(r, R, \phi, \xi),$$

where ξ is the finite sequence of all $rx_1^{-k_1} x_2^{-k_2}$ with $b_{k_1, k_2} \neq 0$ (in any order). Notice that this is only a finite intersection since P is a polynomial. Since $|x_1| = |x_2| = 1$, we see that D has a shape as in Figure 2 and that F has singularities at $t = rx_1^{-k_1} x_2^{-k_2}$. We thus may use Theorem 2.14 and therefore need to take a look at the behavior of F near each singularity. We assumed that $x_1^{k_1} x_2^{k_2} \neq x_1^{m_1} x_2^{m_2}$ for $(m_1, m_2) \neq (k_1, k_2)$, which implies that the singularities are distinct, and thus near the point $rx_1^{-k_1} x_2^{-k_2}$, $F(t)$ has the expansion

$$\begin{aligned} (3.22) \quad F(t) &= b_{k_1, k_2} \vartheta \log \left(\frac{1}{1 - tx_1^{k_1} x_2^{k_2} / r} \right) + b_{k_1, k_2} K \\ &+ \sum_{(m_1, m_2) \neq (k_1, k_2)} b_{m_1, m_2} g_{\Theta}(rx_1^{m_1 - k_1} x_2^{m_2 - k_2}) + O(t - rx_1^{-k_1} x_2^{-k_2}) \end{aligned}$$

for $t \rightarrow rx_1^{-k_1} x_2^{-k_2}$. This shows that we can apply Theorem 2.14. Combining this together with Lemma 2.11 proves the theorem. \square

REMARK. For simplicity we have assumed that all the singularities are distinct. The modification required to cope with the case when $x_1^{k_1} x_2^{k_2} = x_1^{m_1} x_2^{m_2}$ for

some $(m_1, m_2) \neq (k_1, k_2)$ would appear in (3.22), but technically there is no restriction. Such a situation, with all the details written out explicitly, appears in [5].

To illustrate this theorem, we will calculate the autocorrelation of two characteristic polynomials at distinct points x_1, x_2 on the unit circle subject to $x_1^{k_1} \neq x_2^{k_2}$ for all $\{k_1, k_2\} \neq \{0, 0\}$.

The four coefficients of $Z_N(x_1)Z_N(x_2)$ are easy to calculate, being $b_{0,0} = b_{1,1} = 1$ and $b_{1,0} = b_{0,1} = -1$ and this enables an immediate simplification to occur by observing that only the terms with b_{k_1,k_2} maximal contribute; the others are of lower order, in this case being $O(N^{-2\vartheta})$.

Substituting these values into the theorem we have

$$(3.23) \quad \begin{aligned} \mathbb{E}_\Theta[Z_N(x_1)Z_N(x_2)] \\ = E_2(0, 0) + E_2(1, 1)x_1^N x_2^N + O\left(\frac{1}{N}\right) + O\left(\frac{1}{N^{2\vartheta}}\right) \end{aligned}$$

with

$$(3.24) \quad E_2(0, 0) = \frac{G_\Theta(rx_1x_2)}{G_\Theta(rx_1)G_\Theta(rx_2)}$$

and

$$(3.25) \quad E_2(1, 1) = \frac{G_\Theta(rx_1^{-1}x_2^{-1})}{G_\Theta(rx_1^{-1})G_\Theta(rx_2^{-1})}.$$

4. Traces.

4.1. *Traces of permutation matrices.* In this section we consider the asymptotic behavior of traces of permutation matrices. Powers of traces and traces of powers have received much attention in the random matrix literature (see, e.g., [6–8]). More specifically, we first look at $\text{Tr}(\sigma^d)$ for fixed $d \in \mathbb{Z}$. Since the embedding of \mathfrak{S}_N into the unitary group in (3.1) is a group homomorphism, we can interpret σ^d as d -fold matrix multiplication and as the matrix corresponding to $\underbrace{\sigma \circ \cdots \circ \sigma}_{d \text{ times}}$.

We first recall a well-known explicit expression for $\text{Tr}(\sigma^d)$ that we shortly prove for completeness.

LEMMA 4.1. *We have for $d \in \mathbb{Z}$*

$$(4.1) \quad \text{Tr}(\sigma^d) = \sum_{k=1}^N \mathbb{1}_{k|d} k C_k(\sigma), \quad \text{with } \mathbb{1}_{k|d} = \begin{cases} 1, & \text{if } k \text{ divides } d, \\ 0, & \text{otherwise.} \end{cases}$$

PROOF. The matrix corresponding to σ^d has the form $(\delta_{i, \sigma^d(j)})$. We thus have

$$(4.2) \quad \text{Tr}(\sigma^d) = \sum_{i=1}^N \delta_{i, \sigma^d(i)} = \#\{i : \sigma^d(i) = i\}.$$

Therefore, $\text{Tr}(\sigma^d)$ is the number of 1-cycles of σ^d . A simple computation now shows that the number of 1-cycles of σ^d is indeed $\sum_{k=1}^N \mathbb{1}_{k|d} k C_k(\sigma)$. \square

Using this expression and the method of generating functions developed in Section 2.2, we prove a weak convergence result for $\text{Tr}(\sigma^d)$.

THEOREM 4.2. *Let $d \in \mathbb{N}$ be given. We then have*

$$(4.3) \quad \sum_{N=0}^{\infty} h_N \mathbb{E}_{\Theta}[e^{is \text{Tr}(\sigma^d)}] t^N = \exp\left(\sum_{k|d} \frac{\theta_k}{k} (e^{isk} - 1) t^k\right) G_{\Theta}(t).$$

If g_{Θ} is of class $\mathcal{F}(\vartheta, r, K)$, then

$$(4.4) \quad \mathbb{E}_{\Theta}[e^{is \text{Tr}(\sigma^d)}] = \exp\left(\sum_{k|d} \frac{\theta_k}{k} (e^{isk} - 1) r^k\right) + o\left(\frac{1}{N}\right).$$

If g_{Θ} is of class $e\mathcal{F}(\vartheta, r, \gamma)$, then

$$(4.5) \quad \mathbb{E}_{\Theta}[e^{is \text{Tr}(\sigma^d)}] = \exp\left(\sum_{k|d} \frac{\theta_k}{k} (e^{isk} - 1) r^k\right) + o\left(\frac{\log(N)}{N^{\gamma}}\right).$$

PROOF. Applying Lemma 4.1, and evaluating the expectation explicitly in terms of partitions using Lemma 2.3, we have

$$(4.6) \quad \sum_{N=0}^{\infty} h_N \mathbb{E}_{\Theta}[e^{is \text{Tr}(\sigma^d)}] t^N = \sum_{\lambda} \frac{1}{z_{\lambda}} \prod_{k=1}^{\infty} (\theta_k e^{isk \mathbb{1}_{k|d}} t^k)^{C_k}.$$

The cycle index theorem (Lemma 2.6) yields that this equals

$$(4.7) \quad \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} \theta_k e^{isk \mathbb{1}_{k|d}} t^k\right) = \exp\left(\sum_{k|d} \frac{\theta_k}{k} (e^{isk} - 1) t^k\right) G_{\Theta}(t),$$

where $G_{\Theta}(t)$ is given in (2.6). This proves equation (4.3).

Applying Theorem 2.10 to this yields equation (4.4), and Theorem 2.16 yields equation (4.5), as required. \square

REMARK. An alternative way to prove Theorem 4.2 is to use Theorem 3.1 in [23], which computes the generating function of $h_N \mathbb{E}_{\Theta}[\exp(\sum_{k=1}^b is_k C_k)]$ and its asymptotic behavior for $g_{\Theta}(t) \in \mathcal{F}(\vartheta, r, K)$ and $g_{\Theta}(t) \in e\mathcal{F}(\vartheta, r, \gamma)$.

We obtain the following as an immediate corollary.

COROLLARY 4.3. Let $d \in \mathbb{Z}$ be fixed and assume that g_Θ is in $\mathcal{F}(\vartheta, r, K)$ or $e\mathcal{F}(\vartheta, r, \gamma)$. Then

$$(4.8) \quad \text{Tr}(\sigma^d) \xrightarrow{d} \sum_{k|d} k P_k \quad \text{as } N \rightarrow \infty,$$

where P_k are independent Poisson distributed random variables with $\mathbb{E}[P_k] = \frac{\theta_k}{k} r^k$.

4.2. *Traces of functions.* Recall from the [Introduction](#) that if M is the permutation matrix representing the permutation σ , then for a function $F: S^1 \rightarrow \mathbb{C}$, we defined the trace of F to be the function $\text{Tr}(F): \mathfrak{S}_N \rightarrow \mathbb{C}$ with

$$(4.9) \quad \text{Tr}(F)(\sigma) := \sum_{k=1}^N F(\omega_k),$$

where $(\omega_k)_{k=1}^N$ are the eigenvalues of M or σ with multiplicity. Lemma 1.2 showed that $\text{Tr}(F)$ could be expressed in terms of the cycle structure of σ as

$$(4.10) \quad \text{Tr}(F)(\sigma) = \sum_{k=1}^N k C_k \Delta_k(F)$$

with

$$(4.11) \quad \Delta_k(F) := \frac{1}{k} \sum_{m=1}^k F(e^{2\pi i m/k}).$$

The asymptotic behavior of $\text{Tr}(F)$ is not so easy to compute for an arbitrary function defined on the unit circle. This problem will be dealt with more carefully in Sections 5 and 6. However, if F is a Laurent polynomial, we can use the same method as for $\text{Tr}(\sigma^d)$.

THEOREM 4.4. Let

$$(4.12) \quad F(x) = \sum_d b_d x^d$$

be a Laurent polynomial. If $g_\Theta \in \mathcal{F}(\vartheta, r, K)$ or $g_\Theta \in e\mathcal{F}(\vartheta, r, \gamma)$, then

$$(4.13) \quad \text{Tr}(F)(\sigma) - N b_0 \xrightarrow{d} \sum_{\substack{d=-\infty \\ d \neq 0}}^{\infty} b_d \sum_{\substack{k \geq 1 \\ k|d}} k P_k \quad \text{as } N \rightarrow \infty,$$

where P_k are independent Poisson distributed random variables with $\mathbb{E}[P_k] = \frac{\theta_k}{k} r^k$.

PROOF. Due to the linearity of $\text{Tr}(F)$, we may assume the constant term, b_0 , is zero. As in the previous computations, we apply the cycle index theorem to obtain

$$(4.14) \quad \sum_{N=0}^{\infty} h_N \mathbb{E}_{\Theta}[e^{is \text{Tr}(F)(\sigma)}] t^N = \sum_{\lambda} \frac{1}{z_{\lambda}} \prod_{k=1}^{\infty} (\theta_k e^{isk \Delta_k(F)} t^k)^{C_k}$$

$$(4.15) \quad = \exp \left(\sum_{k=1}^{\infty} \frac{\theta_k}{k} e^{isk \Delta_k(F)} t^k \right)$$

and since $\Delta_k(x^d) = \mathbb{1}_{k|d}$ and is linear, this equals

$$(4.16) \quad \exp \left(\sum_{k=1}^{\infty} \frac{\theta_k}{k} \exp \left(is \sum_{d \neq 0} b_d k \mathbb{1}_{k|d} \right) t^k \right) \\ = \exp \left(\sum_{k=1}^{\infty} \frac{\theta_k}{k} \left(\prod_{\substack{d \neq 0 \\ k|d}} e^{is b_d k} - 1 \right) t^k \right) G_{\Theta}(t).$$

Note the first factor is entire, so Theorem 2.10 [for the case of $\mathcal{F}(r, \vartheta, K)$] and Theorem 2.16 [for the case of $e\mathcal{F}(r, \vartheta, \gamma)$] yields

$$(4.17) \quad \mathbb{E}_{\Theta}[e^{is \text{Tr}(F)(\sigma)}] \rightarrow \exp \left(\sum_{k=1}^{\infty} \frac{\theta_k}{k} \left(\prod_{\substack{d \neq 0 \\ k|d}} e^{is b_d k} - 1 \right) r^k \right).$$

The right-hand side is the characteristic function of the right-hand side in (4.13). The proof is complete. \square

THEOREM 4.5. *Let $F : S^1 \rightarrow \mathbb{C}$ be of bounded variation, and $d \in \mathbb{N}$ be given. Then*

$$(4.18) \quad \frac{1}{N^d} \mathbb{E}_{\Theta}[(\text{Tr}(F)(\sigma))^d] = \left(\int_{S^1} F(\varphi) d\varphi \right)^d + O\left(\frac{\mathbb{E}_{\Theta}[T(\sigma)]}{N}\right),$$

where $T(\sigma)$ is the total number of cycles of σ , and $d\varphi$ the uniform measure on S^1 .

Moreover, if $g_{\Theta}(t) \in \mathcal{F}(\vartheta, r, K)$ or $g_{\Theta}(t) \in e\mathcal{F}(\vartheta, r, \gamma)$, then $\mathbb{E}_{\Theta}[T(\sigma)] \sim \vartheta \log(N)$, and thus we have a quick convergence of the moments.

PROOF. Since F is of bounded variation we can apply Koksma's inequality ([19], Theorem 5.1) to see that

$$(4.19) \quad \left| \frac{1}{k} \sum_{m=1}^k F(e^{2\pi i m/k}) - \int_{S^1} F(\varphi) d\varphi \right| \leq 2D_k V(F)$$

with $V(F)$ the variation of F and D_k the discrepancy of the sequence $(e^{2\pi im/k})_{m=1}^k$. But the discrepancy D_k is dominated by $1/k$. We thus have

$$(4.20) \quad \Delta_k(F) = \frac{1}{k} \sum_{m=1}^k F(e^{2\pi im/k}) = \int_{S^1} F(\varphi) d\varphi + O\left(\frac{1}{k}\right).$$

We now combine (4.10) and (4.20) and get

$$(4.21) \quad \begin{aligned} \text{Tr}(F)(\sigma) &= \sum_{k=1}^N C_k \left(k \int_{S^1} F(\varphi) d\varphi + O(1) \right) \\ &= N \int_{S^1} F(\varphi) d\varphi + O(T(\sigma)), \end{aligned}$$

where we have used that $\sum_{k=1}^N k C_k = N$ and $\sum_{k=1}^N C_k = T(\sigma)$. Notice that (4.21) is independent of any probability measure on \mathfrak{S}_N . Using the binomial theorem and the fact that $0 < T(\sigma)/N \leq 1$ for all σ , we get

$$(4.22) \quad \frac{1}{N^d} (\text{Tr}(F)(\sigma))^d = \left(\int_{S^1} F(\varphi) d\varphi \right)^d + O_{F,d} \left(\frac{T(\sigma)}{N} \right),$$

where the constant implicit in the big-O is independent of σ and N . We apply $\mathbb{E}_\Theta[\cdot]$ on both sides, and this proves the first part of the theorem.

The last statement follows from [23], Theorem 4.2, where it is shown that if $g_\Theta(t) \in \mathcal{F}(\vartheta, r, K)$ or $g_\Theta(t) \in e\mathcal{F}(\vartheta, r, \gamma)$ then $\mathbb{E}_\Theta[T(\sigma)] \sim \vartheta \log(N)$. \square

REMARK. In fact, for many probability distributions on \mathfrak{S}_N , $\mathbb{E}_\Theta[T(\sigma)] = o(N)$. The only way for this not to be true is for σ to frequently have only small cycles, which will occur if $\Theta = (\theta_k)_{k=1}^\infty$ is a sequence tending to zero very rapidly.

5. Wreath product, traces and the generalized Ewens measure. In this section we consider the traces of the wreath product $S^1 \wr \mathfrak{S}_N$ (see, e.g., [27]). More precisely, we consider random matrices of the form

$$M(\sigma, z_1, \dots, z_N) := \text{diag}(z_1, \dots, z_N) \cdot \sigma,$$

where σ is a random permutation of \mathfrak{S}_N , and $(z_j)_{j \geq 1}$ is a sequence of i.i.d. random variables with values in S^1 (the complex unit circle), independent of σ . Many groups closely related to \mathfrak{S}_N give similar matrices, for instance, the Weyl group of $\text{SO}(2N)$.

The trace of a function F is then extended in the obvious way by

$$(5.1) \quad \text{Tr}(F) = \text{Tr}(F, z_1, \dots, z_N)(\sigma) := \sum_{k=1}^N F(\omega_k),$$

where $(\omega_k)_{k=1}^N$ are the N eigenvalues of $M(\sigma, z_1, \dots, z_N)$.

We now give a more explicit expression of $\text{Tr}(F)$.

LEMMA 5.1.

$$(5.2) \quad \text{Tr}(F) \stackrel{d}{=} \sum_{k=1}^N \sum_{m=1}^{C_k} k \Delta_k(F, Z_{k,m}),$$

where $(Z_{k,m})_{k,m \geq 1}$ is a sequence of independent random variables which is independent of $(C_k)_{k \geq 1}$ (the sequence of cycle numbers of σ), with $Z_{k,m}$ equal in distribution to $\prod_{j=1}^k z_j$, and

$$(5.3) \quad \Delta_k(F, y) := \frac{1}{k} \sum_{\omega^k=y} F(\omega).$$

PROOF. The characteristic polynomial of $M(\sigma, z_1, \dots, z_N)$ with $\sigma \in \mathfrak{S}_N$ with cycle type λ , is given by

$$(5.4) \quad \det(1 - x M(\sigma, z_1, \dots, z_N)) = \prod_{k=1}^N \prod_{m=1}^{C_k} \left(1 - x^k \prod_{j=1}^k z_j^{k,m} \right),$$

where the sequence $(z_j^{k,m})_{k,m,j}$ is the same sequence as $(z_j)_{j=1}^N$, but with a different numeration and ordering. [Note that this is why it is crucial that the (z_j) are i.i.d.] The proof of (5.4) is similar to the proof of (3.3) and we thus omit the details. The lemma now follows immediately from (5.4). \square

As in Section 4, we can compute the generating function of $\text{Tr}(F)$.

LEMMA 5.2. We define

$$(5.5) \quad \chi_k(s) := \mathbb{E}[e^{isk \Delta_k(F, Z_{k,m})}].$$

We then have

$$(5.6) \quad \sum_{N=0}^{\infty} h_N \mathbb{E}_{\Theta}[e^{is \text{Tr}(F)}] t^N = \exp\left(\sum_{k=1}^{\infty} \frac{\theta_k}{k} \chi_k(s) t^k\right).$$

REMARK. Note that $\chi_k(s)$ is independent of m since $Z_{k,1} \stackrel{d}{=} Z_{k,2} \stackrel{d}{=} \dots \stackrel{d}{=} Z_{k,m}$.

PROOF OF LEMMA 5.2. We compute $\mathbb{E}[\exp(is \text{Tr}(F))]$. For this we use the independence of C_k and Δ_k to obtain

$$(5.7) \quad \begin{aligned} \mathbb{E}[e^{is \text{Tr}(F)}] &= \mathbb{E}\left[\prod_{k=1}^N \prod_{m=1}^{C_k} e^{isk \Delta_k(F, Z_{k,m})}\right] = \mathbb{E}\left[\prod_{k=1}^N \prod_{m=1}^{C_k} \chi_k(s)\right] \\ &= \mathbb{E}\left[\prod_{k=1}^N (\chi_k(s))^{C_k}\right]. \end{aligned}$$

The theorem now follows immediately from Lemma 2.6. \square

DEFINITION 5.3. Let

$$(5.8) \quad g_{\text{Tr}(F)}(t) := \sum_{k=1}^{\infty} \frac{\theta_k}{k} \chi_k(s) t^k.$$

THEOREM 5.4. Assume $\mathbb{E}[|\Delta_k(F, Z_{k,1})|] = O(k^{-1-\delta})$ for some $0 < \delta \leq 1$, and assume $g_{\Theta}(t)$ is in $e\mathcal{F}(r, \vartheta, \gamma)$, where $e\mathcal{F}(r, \vartheta, \gamma)$ is given in Definition 2.15. Then $g_{\text{Tr}(F)}(t)$ is in $e\mathcal{F}(r, \vartheta, \min\{\gamma, \delta\})$ and

$$(5.9) \quad \mathbb{E}_{\Theta}[e^{is \text{Tr}(F)}] = \exp\left(\sum_{k=1}^{\infty} \frac{\theta_k}{k} (\chi_k(s) - 1) r^k\right) + O(N^{-\min\{\gamma, \delta\}} \log(N))$$

and as N tends to infinity, $\text{Tr}(F)$ converges in law to the random variable

$$(5.10) \quad Y := \sum_{k=1}^{\infty} \sum_{m=1}^{P_k} k \Delta_k(F, Z_{k,m}),$$

where $(P_k)_{k \geq 1}$ is a sequence of independent Poisson random variables, independent of $(Z_{k,m})_{k,m \geq 1}$, and such that P_k has parameter $\theta_k r^k / k$. Here, the series defining Y is a.s. absolutely convergent.

REMARK. By linearity of trace, if F is Riemann integrable one can always subtract a suitable constant to make $\int_{S^1} F(\varphi) d\varphi = 0$, which ensures $\Delta_k(F, z) \rightarrow 0$ as $k \rightarrow \infty$.

REMARK. One should compare equation (5.10) with equation (5.2). The replacement of the cycle counts C_k with P_k is indicative of Feller coupling for the generalized Ewens measure.

PROOF OF THEOREM 5.4. We have

$$(5.11) \quad g_{\text{Tr}(F)}(t) = \sum_{k=1}^{\infty} \frac{\theta_k}{k} \chi_k(s) t^k = g_{\Theta}(t) + \sum_{k=1}^{\infty} \frac{\theta_k}{k} (\chi_k(s) - 1) t^k.$$

We now have

$$(5.12) \quad |\chi_k(s) - 1| \leq \mathbb{E}[|e^{isk \Delta_k(F, Z_{k,1})} - 1|] \leq \mathbb{E}[(k |\Delta_k(F, Z_{k,1})|)] = O(k^{-\delta}).$$

On the other hand, we have $\theta_k = O(r^{-k})$. This follows immediately from the fact that $g_{\Theta}(t)$ is in $e\mathcal{F}(r, \vartheta, \gamma)$. We thus have

$$(5.13) \quad \frac{\theta_k}{k} (\chi_k(s) - 1) = O(r^{-k} k^{-1-\delta}).$$

This shows that $g_{\text{Tr}(F)}(t) \in e\mathcal{F}(r, \vartheta, \min\{\gamma, \delta\})$.

Since $g_{\Theta}(t) \in e\mathcal{F}(r, \vartheta, \gamma)$, we can write $g_{\Theta}(t) = \vartheta \log(\frac{1}{1-t/r}) + g_0(t)$ with $g_0(r) < \infty$. Thus,

$$(5.14) \quad g_{\text{Tr}(F)}(t) = \vartheta \log\left(\frac{1}{1-t/r}\right) + g_0(t) + \sum_{k=1}^{\infty} \frac{\theta_k}{k} (\chi_k(s) - 1)t^k.$$

We get with Theorem 2.16 that

$$(5.15) \quad h_N \mathbb{E}_{\Theta}[e^{is \text{Tr}(F)}] = \frac{N^{\vartheta-1}}{r^N \Gamma(\vartheta)} \exp\left(g_0(r) + \sum_{k=1}^{\infty} \frac{\theta_k}{k} (\chi_k(s) - 1)r^k\right) + R_N$$

with

$$(5.16) \quad R_N = O\left(\frac{N^{\vartheta-1-\min\{\gamma, \delta\}} \log(N)}{r^N}\right).$$

Dividing by h_N proves equation (5.9).

Using the characteristic function of $\text{Tr}(F)$, we can deduce its convergence in law to Y . The absolute convergence of the series in (5.10) comes from

$$(5.17) \quad \begin{aligned} \mathbb{E}\left[\sum_{k=1}^{\infty} \sum_{m=1}^{P_k} k |\Delta_k(F, Z_{k,m})|\right] &= \sum_{k=1}^{\infty} k \mathbb{E}[P_k] \mathbb{E}[|\Delta_k(F, Z_{k,1})|] \\ &= \sum_{k \geq 1} \theta_k r^k O(k^{-1-\delta}) \\ &< \infty, \end{aligned}$$

since $\theta_k = O(r^{-k})$. Now, for $s \in \mathbb{R}$ and $k \geq 1$,

$$(5.18) \quad \begin{aligned} \mathbb{E}[e^{isk \sum_{m=1}^{P_k} \Delta_k(F, Z_{k,m})}] &= \mathbb{E}[(\mathbb{E}[e^{isk \Delta_k(F, Z_{k,1})}])^{P_k}] = \mathbb{E}[(\chi_k(s))^{P_k}] \\ &= \exp\left(\frac{\theta_k}{k} (\chi_k(s) - 1)r^k\right). \end{aligned}$$

Hence, by absolute convergence,

$$(5.19) \quad \mathbb{E}[e^{isY}] = \exp\left(\sum_{k=1}^{\infty} \frac{\theta_k}{k} (\chi_k(s) - 1)r^k\right),$$

and thus by equation (5.9), $\mathbb{E}_{\Theta}[e^{is \text{Tr}(F)}] \rightarrow \mathbb{E}[e^{isY}]$ as $N \rightarrow \infty$, and thus $\text{Tr}(F)$ converges in law to Y . \square

With a more direct approach one can prove the convergence in law of $\text{Tr}(F)$ to Y (albeit without a rate of convergence) under slightly weaker conditions.

THEOREM 5.5. *Assume that $g_{\Theta}(t)$ is in $e\mathcal{F}(r, \vartheta, \gamma)$ and that*

$$(5.20) \quad \sum_{k=1}^{\infty} k^{(1-\vartheta)+} \mathbb{E}[|\Delta_k(F, Z_{k,1})|] < \infty.$$

Then $\text{Tr}(F)$ converges in law to Y , where Y is given by (5.10).

PROOF. Under these conditions, the absolute convergence of the series defining Y is checked as follows:

$$(5.21) \quad \mathbb{E} \left[\sum_{k=1}^{\infty} \sum_{m=1}^{P_k} k |\Delta_k(F, Z_{k,m})| \right] = \sum_{k=1}^{\infty} k \mathbb{E}[P_k] \mathbb{E}[|\Delta_k(F, Z_{k,1})|]$$

$$(5.22) \quad = \sum_{k=1}^{\infty} \theta_k r^k \mathbb{E}[|\Delta_k(F, Z_{k,1})|]$$

$$(5.23) \quad = O \left(\sum_{k=1}^{\infty} \mathbb{E}[|\Delta_k(F, Z_{k,1})|] \right)$$

and this converges by assumption.

In [23], Corollary 3.1.1, it is proven that for all fixed $b \geq 1$, (C_1, C_2, \dots, C_b) tends in law to (P_1, P_2, \dots, P_b) when the dimension N goes to infinity.

Now let

$$(5.24) \quad \text{Tr}_b(F) := \sum_{k=1}^b \sum_{m=1}^{C_k} k \Delta_k(F, Z_{k,m})$$

and

$$(5.25) \quad Y_b := \sum_{k=1}^b \sum_{m=1}^{P_k} k \Delta_k(F, Z_{k,m}).$$

The same argumentation as in Theorem 5.4 gives

$$(5.26) \quad \sum_{N=0}^{\infty} h_N \mathbb{E}_{\Theta} [e^{is \text{Tr}_b(F)}] t^N = \exp \left(\sum_{k=1}^b \frac{\theta_k}{k} (\chi_k(s) - 1) t^k \right) e^{g_{\Theta}(t)}$$

from which follows (again by the same reasoning as in Theorem 5.4)

$$(5.27) \quad \mathbb{E}_{\Theta} [e^{is \text{Tr}_b(F)}] \rightarrow \mathbb{E} [e^{is Y_b}]$$

as $N \rightarrow \infty$. Here b is fixed but arbitrary, so this convergence implies (by using the inequality $|e^{ix} - e^{iy}| \leq |x - y|$) that

$$(5.28) \quad \begin{aligned} & \limsup_{N \rightarrow \infty} |\mathbb{E}_{\Theta} [e^{is \text{Tr}(F)}] - \mathbb{E} [e^{is Y}]| \\ & \leq |s| \limsup_{N \rightarrow \infty} \sum_{k=b+1}^{\infty} k \mathbb{E}[|\Delta_k(F, Z_{k,1})|] \mathbb{E}'[(C_k + P_k)] \\ & \leq |s| \sum_{k=b+1}^{\infty} k \mathbb{E}[|\Delta_k(F, Z_{k,1})|] H_k, \end{aligned}$$

where \mathbb{E}' is the expectation over the product measure of \mathbb{P}_Θ and the measures occurring from $(P_k)_{k \geq 1}$, and where

$$(5.29) \quad H_k = \mathbb{E}[P_k] + \sup_{N \geq 1} \mathbb{E}_\Theta[C_k].$$

Therefore, the theorem is proven if we show that

$$(5.30) \quad \sum_{k=1}^{\infty} k H_k \mathbb{E}[|\Delta_k(F, Z_{k,1})|] < \infty.$$

Ercolani and Ueltschi [9], Proposition 2.1(c), show that

$$(5.31) \quad \mathbb{E}_\Theta[C_k] = \begin{cases} \frac{\theta_k}{k} \frac{h_{N-k}}{h_N}, & \text{if } k \leq N, \\ 0, & \text{if } k > N. \end{cases}$$

By Lemma 2.17, we have, for some $A > 0$ and for N going to infinity,

$$(5.32) \quad h_N \sim A(N+1)^{\vartheta-1}/r^N,$$

so

$$(5.33) \quad \frac{h_{N-k}}{h_N} = O\left(r^k \left(1 - \frac{k}{N+1}\right)^{\vartheta-1}\right).$$

Now, for k fixed,

$$(5.34) \quad \begin{aligned} & \max_{N \geq k} \left(1 - \frac{k}{N+1}\right)^{\vartheta-1} \\ &= \begin{cases} 1, & \text{if } \vartheta \geq 1, \\ (k+1)^{1-\vartheta}, & \text{if } \vartheta < 1 \text{ (attained at } N=k). \end{cases} \end{aligned}$$

Since $g_\Theta(t) \in e\mathcal{F}(r, \vartheta, \gamma)$, we have $\theta_k r^k = O(1)$, and so we deduce that

$$(5.35) \quad \sup_{N \geq 1} \mathbb{E}_\Theta[C_k] = O\left(\frac{\theta_k}{k} r^k (k+1)^{(1-\vartheta)_+}\right) = O(k^{-1+(1-\vartheta)_+}).$$

Finally, since

$$(5.36) \quad \mathbb{E}[P_k] = \theta_k r^k / k = O(1/k)$$

we have

$$(5.37) \quad H_k = O(k^{-1+(1-\vartheta)_+}),$$

and so

$$(5.38) \quad \sum_{k=1}^{\infty} k H_k \mathbb{E}[|\Delta_k(F, Z_{k,1})|] = O\left(\sum_{k=1}^{\infty} k^{(1-\vartheta)_+} \mathbb{E}[|\Delta_k(F, Z_{k,1})|]\right) < \infty$$

as required. \square

REMARK. In the case when z_i are all equal to 1 almost surely, then $\Delta_k(F, Z_{k,m}) = \Delta_k(F)$ as given in (4.11), and we are back in the case of permutation matrices. Thus these two theorems fulfill the promise made in Section 4.

The following result gives sufficient conditions, expressed only in terms of the Fourier coefficients of F , under which we can be assured the conditions of Theorems 5.4 and 5.5 apply.

THEOREM 5.6. *Let us suppose that F is continuous and for $m \in \mathbb{Z}$, let us define the m th Fourier coefficient of F by*

$$(5.39) \quad c_m(F) := \frac{1}{2\pi} \int_0^{2\pi} e^{-imx} F(e^{ix}) dx.$$

We assume that the mean value of F vanishes, that is, $c_0(F) = 0$. If for some $\delta \in (0, 1]$, $c_m(F) = O(|m|^{-1-\delta})$ when $|m|$ goes to infinity then

$$(5.40) \quad \mathbb{E}[|\Delta_k(F, Z_{k,1})|] = O(k^{-1-\delta}).$$

If there exists $s > (1 - \vartheta)_+$ such that

$$(5.41) \quad \sum_{m \in \mathbb{Z}} |m|^s |c_m(F)| < \infty$$

then

$$(5.42) \quad \sum_{k=1}^{\infty} k^{(1-\vartheta)_+} \mathbb{E}[|\Delta_k(F, Z_{k,1})|] < \infty.$$

REMARK. If the assumptions of Theorem 5.6 are satisfied, except that $c_0(F) = 0$, then one can still apply the result to the function $F - c_0(F)$, and deduce, from Theorem 5.4 or Theorem 5.5, that $\text{Tr}(F) - Nc_0(F)$ converges in law to Y , where Y is given by (5.10).

PROOF OF THEOREM 5.6. Since F is continuous, one has, for all $x \in [0, 2\pi)$,

$$(5.43) \quad F(e^{ix}) = \lim_{n \rightarrow \infty} \sum_{m \in \mathbb{Z}} \frac{(n - |m|)_+}{n} c_m(F) e^{imx},$$

by using the Fejér kernel. Now, by assumption,

$$(5.44) \quad \sum_{m \in \mathbb{Z}} |c_m(F)| < \infty,$$

and hence, by dominated convergence,

$$(5.45) \quad F(e^{ix}) = \sum_{m \in \mathbb{Z}} c_m(F) e^{imx},$$

where the series is absolutely convergent. Since $c_0(F) = 0$, one deduces that for all $k \geq 1$ and $x \in [0, 2\pi)$,

$$\begin{aligned}\Delta_k(F, e^{ix}) &= \frac{1}{k} \sum_{j=0}^{k-1} F(e^{i(x+2j\pi)/k}) \\ &= \frac{1}{k} \sum_{m \in \mathbb{Z} \setminus \{0\}} c_m(F) \left(\sum_{j=0}^{k-1} e^{im(x+2j\pi)/k} \right) \\ &= \sum_{\substack{m \in \mathbb{Z} \setminus \{0\}, \\ k|m}} c_m(F) e^{imx/k}.\end{aligned}$$

If F satisfies the first assumption, that $c_m(F) = O(|m|^{-1-\delta})$, then

$$(5.46) \quad \sup_{x \in [0, 2\pi)} |\Delta_k(F, e^{ix})| \leq \sum_{\substack{m \in \mathbb{Z} \setminus \{0\}, \\ k|m}} |c_m(F)| = O\left(\sum_{\substack{m \in \mathbb{Z} \setminus \{0\}, \\ k|m}} |m|^{-1-\delta} \right)$$

$$(5.47) \quad = O(k^{-1-\delta})$$

for k going to infinity, which clearly implies $\mathbb{E}[|\Delta_k(F, Z_{k,1})|] = O(k^{-1-\delta})$.

If F satisfies the second assumption, one has

$$(5.48) \quad \sum_{k=1}^{\infty} k^{(1-\vartheta)_+} \sup_{x \in [0, 2\pi)} |\Delta_k(F, e^{ix})| \leq \sum_{k=1}^{\infty} k^{(1-\vartheta)_+} \sum_{\substack{m \in \mathbb{Z} \setminus \{0\}, \\ k|m}} |c_m(F)|$$

$$(5.49) \quad \leq \sum_{k=1}^{\infty} \sum_{\substack{m \in \mathbb{Z} \setminus \{0\}, \\ k|m}} |m|^{(1-\vartheta)_+} |c_m(F)|$$

$$(5.50) \quad \leq \sum_{m \in \mathbb{Z} \setminus \{0\}} |c_m(F)| |m|^{(1-\vartheta)_+} \tau(|m|),$$

where $\tau(|m|)$ denotes the number of divisors of $|m|$. Since $\tau(|m|) = O(|m|^\varepsilon)$ for all $\varepsilon > 0$, one deduces that

$$(5.51) \quad \sum_{k \geq 1} k^{(1-\vartheta)_+} \mathbb{E}[|\Delta_k(F, Z_{k,1})|] = O\left(\sum_{m \in \mathbb{Z} \setminus \{0\}} |c_m(F)| |m|^s \right) < \infty.$$

The proof of the theorem is complete. \square

COROLLARY 5.7. *Let F be a continuous function from S^1 to \mathbb{C} , contained in a Sobolev space H^s for some $s > 1/2 + (1 - \vartheta)_+$. Then, the second condition of Theorem 5.6 is fulfilled, and thus also the conditions of Theorem 5.5.*

PROOF. By the Cauchy–Schwarz inequality, one has, for any $\alpha \in ((1 - \vartheta)_+, s - 1/2)$,

$$(5.52) \quad \sum_{m \in \mathbb{Z} \setminus \{0\}} |m|^\alpha |c_m(F)| \leq \left(\sum_{m \in \mathbb{Z} \setminus \{0\}} |m|^{2s} |c_m(F)|^2 \right)^{1/2} \left(\sum_{m \in \mathbb{Z} \setminus \{0\}} |m|^{2(\alpha-s)} \right)^{1/2},$$

which is finite since $F \in H^s$ and $2(\alpha - s) < -1$. \square

REMARK. Note that it is not always obvious to estimate directly the Fourier coefficients of a function F , however, standard results from Fourier analysis concerning the differentiability of F yield sufficient bounds on the decay of the Fourier coefficients of F for the conditions of Theorem 5.6 to be checked (see, e.g. [18], Chapter 9).

6. Diverging variance for the classical Ewens measure. In the previous two sections, we have been considering the convergence of $\text{Tr}(F)$ to some limit for random permutation matrices (and their generalization to wreath products), where the underlying probability space is the generalized Ewens measure. The conditions we have used have all implied that the variance of $\text{Tr}(F)$ stays bounded as $N \rightarrow \infty$.

A recent paper by Ben Arous and Dang [2] dealing with $\text{Tr}(F)$ for real F and for random permutation matrices in the special case of the classical Ewens measure (which is when $\theta_k = \theta$, a constant), demonstrates a dichotomy between converging and diverging variance for $\text{Tr}(F)$ in the classical Ewens distribution. In the former case they also show convergence of $\text{Tr}(F)$ to an explicit finite limit, and in the latter case they prove the following central limit theorem.

THEOREM 6.1 (Ben Arous and Dang). *Let $F: \mathbb{C} \rightarrow \mathbb{R}$ be given and assume that*

$$(6.1) \quad V_N := \theta \sum_{k=1}^N k \Delta_k(F)^2$$

tends to infinity as $N \rightarrow \infty$ and

$$(6.2) \quad \max_{1 \leq k \leq N} k |\Delta_k(F)| = o(\sqrt{V_N})$$

then,

$$(6.3) \quad \frac{\text{Tr}(F) - \mathbb{E}[\text{Tr}(F)]}{\sqrt{V_N}} \xrightarrow{d} \mathcal{N}(0, 1).$$

In the generalized Ewens measure, we are currently unable to apply the function theoretic methods to prove weak convergence results in the case of diverging variance. However, for the classical Ewens measure we are able to prove a similar central limit theorem the wreath product, with slightly extended application, in the sense that condition (6.2) can be weakened from a sup-norm to a p -norm.

THEOREM 6.2. *Let $F : \mathbb{C} \rightarrow \mathbb{R}$ be given and assume that*

$$(6.4) \quad V_N := \theta \sum_{k=1}^N k \mathbb{E}[(\Delta_k(F, Z_{k,1}))^2]$$

tends to infinity as $N \rightarrow \infty$. Assume further that there exists a $p > \max\{\frac{1}{\theta}, 2\}$ such that

$$(6.5) \quad \sum_{k=1}^N k^{p-1} \mathbb{E}[|\Delta_k(F, Z_{k,1})|^p] = o(V_N^{p/2})$$

with $\Delta_k(F, z) = \frac{1}{k} \sum_{\omega^k=z} F(\omega)$. Then

$$(6.6) \quad \left(\frac{\text{Tr}(F) - E_N}{\sqrt{V_N}} \right)_{N \geq 1}$$

converges in distribution to a standard Gaussian random variable, where

$$(6.7) \quad E_N := \theta \sum_{k=1}^N \mathbb{E}[\Delta_k(F, Z_{k,1})].$$

The behavior for complex functions F can be computed in a similar way. We consider here only real F to keep the notation simple and to avoid further technicalities.

REMARK. Recall that without loss of generality we may assume F has mean zero in the sense that $\int_0^{2\pi} F(e^{ix}) dx = 0$. We remark that this does not necessarily imply that $\Delta_k(F, z)$ tends to zero, even though $\Delta_k(F, z)$ is a discretization of the integral, without the assumption of further smoothness conditions. Moreover, note that in the framework of the symmetric group (i.e., $Z_{k,1} = 1$ almost surely), the assumption (6.5) is implied by the condition (6.2) given in [2]. Indeed, if (6.2) is satisfied, then for $p > 2$, one has

$$\begin{aligned} \sum_{k=1}^N k^{p-1} |\Delta_k(F)|^p &\leq \left(\max_{1 \leq k \leq N} (k |\Delta_k(F)|) \right)^{p-2} \sum_{k=1}^N k |\Delta_k(F)|^2 \\ &= o(V_N^{(p-2)/2}) O(V_N) = o(V_N^{p/2}). \end{aligned}$$

In the proof of Theorem 6.2, we use the Feller coupling, which allows the random variables C_k and P_k to be defined on the same space and to replace the weak convergence $C_k \xrightarrow{d} P_k$ by convergence in probability (but not a.s. convergence). This coupling exists only for the classical Ewens measure and thus P_k are independent Poisson distributed random variables with $\mathbb{E}[P_k] = \frac{\theta}{k}$. The construction and further details can be found, for instance, in [1], Sections 1 and 4.

The Feller coupling allows us to prove Theorem 6.2 with C_k replaced by P_k , and the following lemma allows us to estimate the distance between the two.

LEMMA 6.3 (Ben Arous and Dang [2]). *For any $\theta > 0$ there exists a constant $K(\theta)$ depending on θ , such that for every $1 \leq m \leq N$,*

$$(6.8) \quad \mathbb{E}[|C_k - P_k|] \leq \frac{K(\theta)}{N} + \frac{\theta}{N} \Psi_N(k),$$

where

$$(6.9) \quad \Psi_N(k) := \binom{N-k+\theta-1}{N-k} \binom{N+\theta-1}{N}^{-1}.$$

PROOF OF THEOREM 6.2. The main idea of the proof is to define the auxiliary random variable

$$(6.10) \quad Y_N(F) := \sum_{k=1}^N \sum_{m=1}^{P_k} k \Delta_k(F, Z_{k,m})$$

and to show that $\text{Tr}(F)$ and $Y_N(F)$ have the same asymptotic behavior after normalization, and that (again after normalization) $Y_N(F)$ satisfies a central limit theorem.

First, we will show that

$$(6.11) \quad \mathbb{E}[|\text{Tr}(F) - Y_N(F)|] = o((V_N)^{1/2}).$$

We use Lemma 6.3 and that $Z_{k,m}$ are independent of C_k and P_k to get

$$(6.12) \quad \begin{aligned} & \mathbb{E}[|\text{Tr}(F) - Y_N(F)|] \\ & \leq \mathbb{E} \left[\sum_{k=1}^N |C_k - P_k| k \mathbb{E}[|\Delta_k(F, Z_{k,1})|] \right] \\ & \leq \frac{K(\theta)}{N} \sum_{k=1}^N k \mathbb{E}[|\Delta_k(F, Z_{k,1})|] + \frac{\theta}{N} \sum_{k=1}^N k \mathbb{E}[|\Delta_k(F, Z_{k,1})|] \Psi_N(k). \end{aligned}$$

For the first term, we apply Jensen's inequality and condition (6.5) to obtain

$$(6.13) \quad \frac{1}{N} \sum_{k=1}^N k \mathbb{E}[|\Delta_k(F, Z_{k,1})|] \leq \left(\frac{1}{N} \sum_{k=1}^N k^p \mathbb{E}[|\Delta_k(F, Z_{k,1})|^p] \right)^{1/p}$$

$$(6.14) \quad \leq \left(\sum_{k=1}^N k^{p-1} \mathbb{E}[|\Delta_k(F, Z_{k,1})|^p] \right)^{1/p}$$

$$(6.15) \quad = o(V_N^{1/2}).$$

Now we deal with the second term in (6.12). If $\theta \geq 1$ then $\Psi_N(k)$ is bounded by 1, so the same argument as above shows that the second summand is also $o(V_N^{1/2})$ in this case.

If $0 < \theta < 1$, we have to be more careful. A simple computation shows that there exists constants K_1, K_2 such that

$$(6.16) \quad \Psi_N(k) \leq \begin{cases} K_1 \left(1 - \frac{k}{N}\right)^{\theta-1}, & \text{for } k < N, \\ K_2 N^{1-\theta}, & \text{for } k = N \end{cases}$$

and so

$$(6.17) \quad \frac{\theta}{N} \sum_{k=1}^N k \mathbb{E}[|\Delta_k(F, Z_{k,1})|] \Psi_N(k)$$

$$(6.18) \quad \leq \theta K_2 N^{1-\theta} \mathbb{E}[|\Delta_N(F, Z_{N,1})|] \\ + \frac{\theta K_1}{N} \sum_{k=1}^{N-1} k \mathbb{E}[|\Delta_k(F, Z_{k,1})|] \left(1 - \frac{k}{N}\right)^{\theta-1}.$$

Using the value of p given in the conditions of the theorem,

$$(6.19) \quad N^{1-\theta} \mathbb{E}[|\Delta_N(F, Z_{N,1})|] = (N^{p-p\theta} \mathbb{E}[|\Delta_N(F, Z_{N,1})|^p])^{1/p}$$

$$(6.20) \quad \leq \left(\sum_{k=1}^N k^{p-p\theta} \mathbb{E}[|\Delta_k(F, Z_{k,1})|^p] \right)^{1/p}$$

$$(6.21) \quad \leq \left(\sum_{k=1}^N k^{p-1} \mathbb{E}[|\Delta_k(F, Z_{k,1})|^p] \right)^{1/p}$$

since $k^{p-p\theta} \leq k^{p-1}$ (since $p\theta > 1$) and $\mathbb{E}[|\Delta_k(F, Z_{k,1})|^p] \leq \mathbb{E}[|\Delta_k(F, Z_{k,1})|^p]$ (since $p > 1$). Thus, by condition (6.5), this is $o(V_N^{1/2})$.

Hölder's inequality gives

$$(6.22) \quad \frac{1}{N} \sum_{k=1}^{N-1} k \mathbb{E}[|\Delta_k(F, Z_{k,1})|] \left(1 - \frac{k}{N}\right)^{\theta-1} \\ \leq \left(\frac{1}{N} \sum_{k=1}^{N-1} k^p \mathbb{E}[|\Delta_k(F, Z_{k,1})|^p] \right)^{1/p} \left(\frac{1}{N} \sum_{k=1}^{N-1} \left(1 - \frac{k}{N}\right)^{q(\theta-1)} \right)^{1/q}$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

After re-ordering the sum, the second factor is $(N^{q(1-\theta)-1} \sum_{j=1}^{N-1} j^{-q(1-\theta)})^{1/q}$, and if $q(\theta - 1) > -1$ then it is bounded above by a constant. Note that

$$\begin{aligned} (\theta - 1)q > -1 &\iff (1 - \theta) < \frac{1}{q} \iff (1 - \theta) < 1 - \frac{1}{p} \\ &\iff \theta > \frac{1}{p} \end{aligned}$$

and thus condition (6.5) now ensures the existence of a $p > \frac{1}{\theta}$ such that the first factor is $o((V_N)^{1/2})$ and the second factor is bounded. This proves (6.11).

Therefore Slutsky's theorem implies that $Y_N(F)$ and $\text{Tr}(F)$ have the same asymptotic distribution after scaling. Thus it suffices to show that

$$(6.23) \quad \frac{Y_N(F) - E_N}{\sqrt{V_N}}$$

converges in law to a standard Gaussian random variable.

We calculate the mean of $Y_N(F)$ by first taking expectation with respect to $Z_{k,m}$ and then with respect to P_k , to obtain

$$\begin{aligned} (6.24) \quad \mathbb{E}[Y_N(F)] &= \sum_{k=1}^N \mathbb{E} \left[\sum_{m=1}^{P_k} k \mathbb{E}[\Delta_k(F, Z_{k,m})] \right] \\ &= \sum_{k=1}^N \mathbb{E}[P_k] k \mathbb{E}[\Delta_k(F, Z_{k,1})], \end{aligned}$$

where we use the fact that $\mathbb{E}[\Delta_k(F, Z_{k,m})] = \mathbb{E}[\Delta_k(F, Z_{k,1})]$ for all m . Finally, since $\mathbb{E}[P_k] = \theta/k$, we see that $\mathbb{E}[Y_N(F)] = E_N$ as defined in (6.7).

For the variance, since P_k and $Z_{k,m}$ are all independent, one can move the sum outside the variance, to obtain

$$(6.25) \quad \text{Var}(Y_N(F)) = \sum_{k=1}^N \text{Var} \left(\sum_{m=1}^{P_k} k \Delta_k(F, Z_{k,m}) \right).$$

Now, the variance of a sum of random length of i.i.d. random variables is given by the following formula:

$$(6.26) \quad \text{Var} \left(\sum_{m=1}^P X_m \right) = \text{Var}(X_1) \mathbb{E}[P] + \text{Var}(P) \mathbb{E}[X_1]^2,$$

if $(X_m)_{m \geq 1}$ are i.i.d., L^2 random variables, and if P is an L^2 variable, independent of $(X_m)_{m \geq 1}$ (this result can be proved by a straightforward calculation). Letting $X_m = k \Delta_k(F, Z_{k,m})$ and $P = P_k$ and knowing that $\mathbb{E}[P_k] = \text{Var}(P_k) = \theta/k$, we deduce that $\text{Var}(Y_N(F)) = V_N$.

Finally we apply the Lyapunov central limit theorem since $Y_N(F)$ is a sum of independent random variables. We will show that

$$(6.27) \quad \sum_{k=1}^N \mathbb{E} \left[\left| \sum_{m=1}^{P_k} k \Delta_k(F, Z_{k,m}) - \mathbb{E}[k P_k \Delta_k(F, Z_{k,1})] \right|^p \right] \\ \ll \sum_{k=1}^N k^{p-1} \mathbb{E}[|\Delta_k(F, Z_{k,1})|^p]$$

and by condition (6.5), with $p > 2$, this is $o(V_N^{p/2})$ which means $\frac{Y_N(F) - E_N}{\sqrt{V_N}}$ converges in law to a standard Gaussian random variable.

For simplicity, let P be a Poisson random variable with parameter θ/k (we think of k as being large), and let $X_m = k \Delta_k(F, Z_{k,1})$ be i.i.d. random variables with $\mathbb{E}[|X_m|^p]$ finite. Then

$$(6.28) \quad \mathbb{E} \left[\left| \sum_{m=1}^P X_m - \mathbb{E}[P] \mathbb{E}[X_1] \right|^p \right] \\ = \mathbb{E} \left[\left| \sum_{m=1}^P (X_m - \mathbb{E}[X_1]) + (P - \mathbb{E}[P]) \mathbb{E}[X_1] \right|^p \right] \\ \leq \left(\mathbb{E} \left[\left| \sum_{m=1}^P (X_m - \mathbb{E}[X_1]) \right|^p \right]^{1/p} + \mathbb{E}[|P - \mathbb{E}[P]|^p]^{1/p} \mathbb{E}[X_1] \right)^p$$

by the generalized triangle inequality.

Now, for all $p > 1$,

$$(6.29) \quad \mathbb{E}[|P - \mathbb{E}[P]|^p] \leq \mathbb{E}[P^p] \ll \mathbb{E}[P]$$

as $\mathbb{E}[P] \rightarrow 0$, and so the second term in (6.28) is

$$(6.30) \quad \mathbb{E}[|P - \mathbb{E}[P]|^p]^{1/p} \mathbb{E}[X_1] \ll \mathbb{E}[P]^{1/p} \mathbb{E}[X_1]$$

$$(6.31) \quad \ll (\mathbb{E}[P] \mathbb{E}[|X_1|^p])^{1/p}$$

by Hölder's inequality, since $p > 1$.

To bound the first term in (6.28), let $q_n = \mathbb{P}[P = n]$, and note that

$$(6.32) \quad \mathbb{E} \left[\left| \sum_{m=1}^P (X_m - \mathbb{E}[X_1]) \right|^p \right] = \sum_{n=0}^{\infty} q_n \mathbb{E} \left[\left| \sum_{m=1}^n (X_m - \mathbb{E}[X_1]) \right|^p \right]$$

$$(6.33) \quad \leq \sum_{n=0}^{\infty} q_n \left(\sum_{m=1}^n \mathbb{E}[|X_m - \mathbb{E}[X_1]|^p]^{1/p} \right)^p$$

$$(6.34) \quad = \sum_{n=0}^{\infty} q_n n^p \mathbb{E}[|X_1 - \mathbb{E}[X_1]|^p]$$

$$(6.35) \quad = \mathbb{E}[P^p] \mathbb{E}[|X_1 - \mathbb{E}[X_1]|^p]$$

$$(6.36) \quad \ll \mathbb{E}[P] \mathbb{E}[|X_1|^p].$$

Thus,

$$(6.37) \quad \mathbb{E} \left[\left| \sum_{m=1}^P X_m - \mathbb{E}[P] \mathbb{E}[X_1] \right|^p \right] \\ \ll ((\mathbb{E}[P] \mathbb{E}[|X_1|^p])^{1/p} + (\mathbb{E}[P] \mathbb{E}[|X_1|^p])^{1/p})^p \\ \ll \mathbb{E}[P] \mathbb{E}[|X_1|^p].$$

Using $\mathbb{E}[P] = \theta/k$ and $\mathbb{E}[|X_1|^p] = k^p \mathbb{E}[|\Delta_k(F, Z_{k,1})|^p]$, and summing for $k = 1, \dots, N$ we have proven (6.27). By condition (6.5), if $p > 2$, then this is $o(V_N^{p/2})$ which by Lyapunov's theorem means that $\frac{Y_N(F) - E_N}{\sqrt{V_n}}$ converges in law to a standard Gaussian random variable as required. \square

Acknowledgment. A. Nikeghbali would like to thank Gérard Ben Arous for helpful discussions and for pointing out the use of the improved bounds in the Feller coupling.

REFERENCES

- [1] ARRATIA, R., BARBOUR, A. D. and TAVARÉ, S. (2003). *Logarithmic Combinatorial Structures: A Probabilistic Approach*. European Mathematical Society, Zürich. [MR2032426](#)
- [2] BEN AROUS, G. and DANG, K. (2011). On fluctuations of eigenvalues of random permutation matrices. Unpublished manuscript. Available at [arXiv:1106.2108](#).
- [3] BETZ, V., UELTSCHI, D. and VELENIK, Y. (2011). Random permutations with cycle weights. *Ann. Appl. Probab.* **21** 312–331. [MR2759204](#)
- [4] BUMP, D. (2004). *Lie Groups. Graduate Texts in Mathematics* **225**. Springer, New York. [MR2062813](#)
- [5] DEHAYE, P. and ZEINDLER, D. (2009). On averages of randomized class functions on the symmetric groups and their asymptotics. Unpublished manuscript. Available at [arXiv:0911.4038](#).
- [6] DIACONIS, P. and EVANS, S. N. (2001). Linear functionals of eigenvalues of random matrices. *Trans. Amer. Math. Soc.* **353** 2615–2633. [MR1828463](#)
- [7] DIACONIS, P. and GAMBURD, A. (2004/06). Random matrices, magic squares and matching polynomials. *Electron. J. Combin.* **11** Research Paper 2, 26. [MR2120097](#)
- [8] DIACONIS, P. and SHAHSHAHANI, M. (1994). On the eigenvalues of random matrices. *J. Appl. Probab.* **31A** 49–62. [MR1274717](#)
- [9] ERCOLANI, N. M. and UELTSCHI, D. (2011). Cycle structure of random permutations with cycle weights. Unpublished manuscript.
- [10] FLAJOLET, P. and SEDGEWICK, R. (2009). *Analytic Combinatorics*. Cambridge Univ. Press, Cambridge. [MR2483235](#)
- [11] HAAKE, F., KUŚ, M., SOMMERS, H.-J., SCHOMERUS, H. and ŻYCZKOWSKI, K. (1996). Secular determinants of random unitary matrices. *J. Phys. A* **29** 3641–3658. [MR1400168](#)
- [12] HAMBLY, B. M., KEEVASH, P., O'CONNELL, N. and STARK, D. (2000). The characteristic polynomial of a random permutation matrix. *Stochastic Process. Appl.* **90** 335–346. [MR1794543](#)

- [13] HUGHES, C. P., KEATING, J. P. and O'CONNELL, N. (2001). On the characteristic polynomial of a random unitary matrix. *Comm. Math. Phys.* **220** 429–451. [MR1844632](#)
- [14] HWANG, H. (1994). Théorèmes limites pour les structures combinatoires et les fonctions arithmétiques. Ph.D. thesis, École Polytechnique.
- [15] HWANG, H.-K. (1999). Asymptotics of Poisson approximation to random discrete distributions: An analytic approach. *Adv. in Appl. Probab.* **31** 448–491. [MR1724562](#)
- [16] KEATING, J. P. and SNAITH, N. C. (2000). Random matrix theory and L -functions at $s = 1/2$. *Comm. Math. Phys.* **214** 91–110. [MR1794267](#)
- [17] KEATING, J. P. and SNAITH, N. C. (2000). Random matrix theory and $\zeta(1/2 + it)$. *Comm. Math. Phys.* **214** 57–89. [MR1794265](#)
- [18] KÖRNER, T. W. (1988). *Fourier Analysis*. Cambridge Univ. Press, Cambridge. [MR0924154](#)
- [19] KUIPERS, L. and NIEDERREITER, H. (1974). *Uniform Distribution of Sequences*. Wiley, New York. [MR0419394](#)
- [20] MACDONALD, I. G. (1995). *Symmetric Functions and Hall Polynomials*, 2nd ed. Oxford Univ. Press, New York. [MR1354144](#)
- [21] MONTGOMERY, H. L. (1973). The pair correlation of zeros of the zeta function. In *Analytic Number Theory (Proc. Sympos. Pure Math., Vol. XXIV, St. Louis Univ., St. Louis, MO, 1972)* 181–193. Amer. Math. Soc., Providence, RI. [MR0337821](#)
- [22] NAJNUDEL, J. and NIKEGHBALI, A. (2010). The distribution of eigenvalues of randomized permutation matrices. Available at arXiv:[1005.0402](#).
- [23] NIKEGHBALI, A. and ZEINDLER, D. (2011). The generalized weighted probability measure on the symmetric group and the asymptotic behaviour of the cycles. Available at arXiv:[1105.2315](#).
- [24] RAINS, E. M. (1997). High powers of random elements of compact Lie groups. *Probab. Theory Related Fields* **107** 219–241. [MR1431220](#)
- [25] WIEAND, K. (2000). Eigenvalue distributions of random permutation matrices. *Ann. Probab.* **28** 1563–1587. [MR1813834](#)
- [26] WIEAND, K. (2002). Eigenvalue distributions of random unitary matrices. *Probab. Theory Related Fields* **123** 202–224. [MR1900322](#)
- [27] WIEAND, K. (2003). Permutation matrices, wreath products, and the distribution of eigenvalues. *J. Theoret. Probab.* **16** 599–623. [MR2009195](#)
- [28] ZEINDLER, D. (2010). Permutation matrices and the moments of their characteristic polynomial. *Electron. J. Probab.* **15** 1092–1118. [MR2659758](#)

C. HUGHES
 DEPARTMENT OF MATHEMATICS
 UNIVERSITY OF YORK
 YORK YO10 5DD
 UNITED KINGDOM
 E-MAIL: christopher.hughes@york.ac.uk

J. NAJNUDEL
 A. NIKEGHBALI
 INSTITUT FÜR MATHEMATIK
 UNIVERSITÄT ZÜRICH
 WINTERTHURERSTRASSE 190
 8057-ZÜRICH
 SWITZERLAND
 E-MAIL: joseph.najnudel@math.uzh.ch
ashkan.nikeghbali@math.uzh.ch

D. ZEINDLER
 FAKULTÄT FÜR MATHEMATIK
 UNIVERSITÄT BIELEFELD
 SONDERFORSCHUNGSBEREICH 701
 POSTFACH 10 01 31
 33501 BIELEFELD
 GERMANY
 E-MAIL: zeindler@math.uni-bielefeld.de